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Persistence Modules And Barcodes

Bachelor Thesis

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Preface

Persistence modules are at the heart of a new, rapidly developing branch of algebraic topology. They provide us with an algebraic tool to study filtered topological spaces. In their rawest form, *persistence modules* consist of a family $(V_t)_{t\in\mathbb{R}}$ of vector spaces together with morphisms $V_s \to V_t$ for $s \leq t$, which satisfy certain compatibility conditions.

In order to develop intuition for the subject, we will highlight one particular way of forming a filtered family of vector spaces with the aid of Morse Theory. This branch of mathematics concerns itself with the study of smooth functions and their critical points. Given a smooth manifold M together with a smooth function $f : M \to \mathbb{R}$, we define the sublevel sets to be $M^t := f^{-1}((-\infty, t])$. If we then fix a field K and consider the homology of the sublevel sets with coefficients in K, we end up with a persistence module whose vector spaces are $V_t := H(M^t; K)$ together with morphisms $V_s \to V_t$ for $s \leq t$ induced by the inclusions $M^s \hookrightarrow M^t$.

One thing that that makes *persistence modules* so appealing for the analysis of filtered topological spaces is that we can link invariants called *barcodes* to them. *Barcodes* are multisets of intervals, i.e. sets of intervals that take multiplicities into account. We will show that barcodes encode when a homology class enters or leaves the filtration. The following figure illustrates the procedure of assocating a barcode to the height function on a deformed 2-sphere.

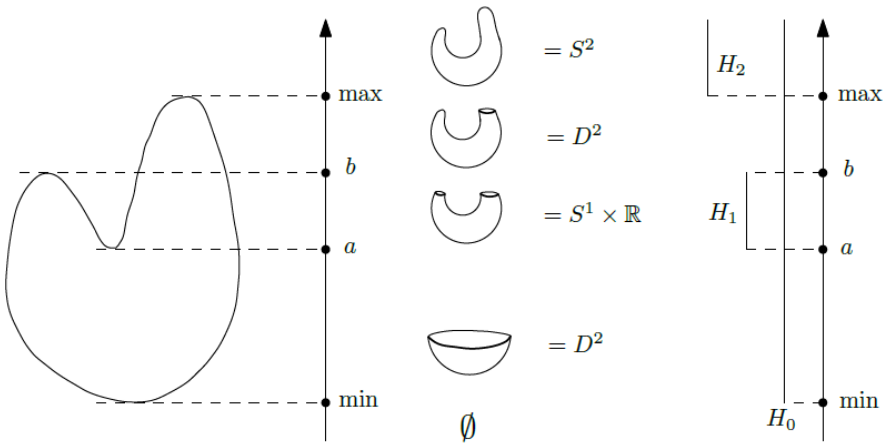


Figure 1: *Left*: The height function on the (topological) 2-sphere. *Middle*: Sublevel sets. *Right*: The corresponding barcode. Figure taken from [7].

We are now going to give an outline of the thesis. In Chapter 1, we will give a brief exposition on Morse theory. In particular, we are going to focus

on showing how the topology of the sublevel sets only changes when the filtration parameter passes a critical value. We will then use this fact to show that we can construct a cell complex that is homotopy equivalent to the manifold that is being filtered, with cells in correspondence with the critical points of the filtration function.

Chapter 2 introduces the category of *persistence modules*. Furthermore we will define an extended pseudometric, the *interleaving distance*, on the space of persistence modules.

Chapter 3 concerns itself with proving the Representation Theorem of Zomorodian and Carlsson, which associates to each persistence module over a field K a graded $K[t]$-module. We then go on to prove a structure theorem for graded modules over graded principal ideals. This theorem in turn allows us to prove the Normal Form Theorem that encodes the algebraic information of a persistence module in terms of multisets of intervals, the *barcodes*.

In Chapter 4, we introduce the *bottleneck distance* on the space of barcodes and prove the Isometry Theorem which shows that the *interleaving distance* between two persistence modules equals the *bottleneck distance* between their associated *barcodes*.

In Chapter 5, we present a few applications of persistence modules. The first example will be to give a lower bound on how well one can approximate a Morse function by another Morse function with exactly two critical points. The second one will bound the length of barcodes of a Morse function on the torus by the L^2-norm of the Laplacian.

Acknowledgement

At this point, the author would like to express his gratitude to Professor Will Merry for supervising this thesis and for introducing him to the subject of persistence modules. His office door was always open for helpful conversations and a multitude of questions were answered on his forum. His passion for teaching is exceptional and is an ongoing source of motivation.

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Chapter 1

Morse Theory

Given a manifold M, Morse Theory provides us with a tool to analyse its topology through the study of a special type of differentiable function

$$f : M \to \mathbb{R},$$

the so-called Morse functions. Intuitively, Morse functions can be thought of as generalised height functions on M. The critical points of Morse functions turn out to have special properties that allow us to retrieve topological information of M.

For example, one result will be that the topology of sublevel sets $f^{-1}((-\infty, t])$ only changes when t passes a critical value of f. This fact can be utilised to *filter* the manifold M and in turn link algebraic invariants to this filtration. We will come back to filtrations induced by Morse functions when we introduce persistence modules in chapter 2.

This chapter is mainly going to follow the exposition given in [2].

1.1 Morse functions

Definition 1.1 (Critical Point) *Let M be a smooth manifold. A **critical point** of a smooth function $f : M \to \mathbb{R}$ is a point p at which*

$$Df(p) : T_pM \to T_{f(p)}\mathbb{R} \cong \mathbb{R} \tag{1.1}$$

vanishes.

Lemma 1.2 *Let $f : M \to \mathbb{R}$ be a smooth map between the smooth manifold M and $\mathbb{R}$. Let $p \in M$, and let $\sigma : U \to O$ be a chart on M about p, and let $\tau : \mathbb{R} \xrightarrow{id} \mathbb{R}$ be the obvious chart on $\mathbb{R}$. Denote the local coordinates of σ by (x^i) and denote the local coordinates of τ by (y^i).*

Then the matrix of $Df(p)$ w.r.t. the bases $\{\frac{\partial}{\partial x^i}|_p \mid i = 1, \dots, n\}$ of T_pM and $\{\frac{\partial}{\partial t}|_t\}$ of $T_{f(p)}\mathbb{R}$ is given by the matrix $D(f \circ \sigma^{-1})(\sigma(x))$.

This lemma gives us an expression in local coordinates of the fact that a point p is critical, namely:

$$D(f \circ \sigma^{-1})(\sigma(x))\big[\frac{\partial}{\partial x^i}|_p\big] = 0 \text{ for all } i. \tag{1.2}$$

Definition 1.3 (Hessian) *Let (M, g) be a smooth Riemannian manifold, and let ∇ denote the Levi-Civita connection. We define the **Hessian of** f to be the tensor of type (0,2) defined by*

$$Hess(f)(X, Y) := X(Y(f)) - df(\nabla_X(Y)).$$

At a critical point p, this expression simplifies to

$$Hess_p(f)(X, Y) = X_p(Y(f))$$

as $df|_p = 0$. Given a chart σ on M around p, denote its local coordinates by (x^i). We have the following expression of the Hessian in local coordinates:

$$Hess_p(f)\big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\big) = \frac{\partial^2(f \circ \sigma^{-1})}{\partial x^i \partial x^j}(\sigma(p)) =: (M_p(f))_{i,j}.$$

It is then straightforward to check that the matrix $M_p(f)$ is symmetric.

Definition 1.4 *Let $f : M \to \mathbb{R}$ be a smooth function.*

- *A critical point $p \in M$ of f is said to be **non-degenerate** if $M_p(f)$ is invertible.*
- *A function $f : M \to \mathbb{R}$ is called a **Morse function** if none of its critical points are degenerate.*
- *The **Morse index** of a critical point $p \in M$ of a Morse function $f : M \to \mathbb{R}$, denoted by $\mu(p)$ is the dimension of a maximal dimensional subspace of T_pM on which $Hess_p(f)$ is negative definite.*

Lemma 1.5 *Non-degenerate critical points are isolated.*

Proof Let $p \in M$ be a non-degenerate critical point of $f : M \to \mathbb{R}$ and let $\sigma : U \subset M \to O \subset \mathbb{R}^n$ be a chart around p such that w.l.o.g. $\sigma(p)$=0. Consider the smooth map

$$\begin{aligned} g : \sigma(U) &\to \mathbb{R}^n \\ x &\mapsto (D_1(f \circ \sigma^{-1})(x), \dots, D_n(f \circ \sigma^{-1})(x)). \end{aligned} \tag{1.3}$$

We have $g(0) = 0$ and $Dg(0) = Hess_0(f \circ \sigma^{-1})$ is non-singular. Thus we can apply the inverse function theorem to obtain a neighbourhood U_0 of 0 to another neighbourhood U_0' of 0 on which g restricts to a diffeomorphism.

In particular, g is injective on U_0, i.e. for all $x \in U_0 \setminus \{0\}$, $g(x) \neq g(0)$. Therefore x is not a critical point of f. □

If we restrict ourselves to compact manifolds, we can say a bit more.

Corollary 1.6 *A Morse function on a compact smooth manifold has a finite number of critical points.*

Proof By the definition of a Morse function every critical point is non-degenerate. Hence by the previous lemma, every critical point is isolated. As any discrete subset of a compact space is finite, the assertion follows. □

Example 1.7 *The prototypical example of a Morse function that one should always keep in mind is the height function on the vertically resting torus .*

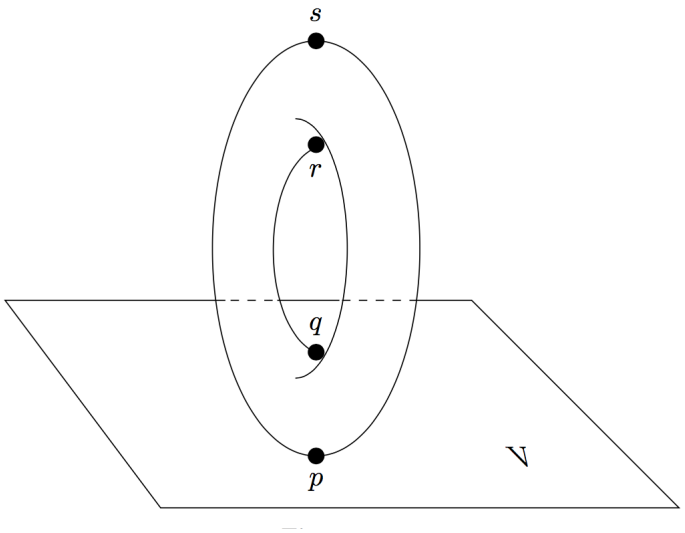


Figure 1.1: taken from [3].

It has one critical point s of index 2, two critical points r and q of index 1 and one critical point p of index 0.

The next lemma gives more insight into how a Morse function behaves locally. However we are not going to prove it and refer to [2, Lemma 3.11] for a proof.

Lemma 1.8 (Morse Lemma) *Let $p \in M$ be a non-degenerate critical point of a smooth function $f : M^n \to \mathbb{R}$ of index k. There exists a smooth chart $\sigma : U \to \mathbb{R}^n$,*

where U is an open neighbourhood of p, with $\sigma(p) = 0$ such that if $\sigma(x) = (x_1, \dots, x_n)$ for $x \in U$, then

$$(f \circ \sigma^{-1})(x_1, \dots, x_n) = f(p) - x_1^2 - x_2^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2.$$

This lemma shows that one can locally interpret the index as being the number of directions in which the manifold curves 'downwards'.

1.2 The gradient flow of a Morse function

In this section, we analyse the flow lines of a special vector field induced by Morse function. To start off, we state an alternative way to see differential 1-forms, i.e. smooth section of the cotangent bundle.

Proposition 1.9 *Let M be a smooth manifold. A $C^\infty(M)$-linear map $\omega : \mathfrak{X}(M) \to C^\infty(M)$ is the same thing as a differential 1-form on M, i.e. an element of $\Omega^1(M)$.*

A proof can be found in [4] Corollary 16.29 therein.

Recall that a Riemannian manifold is defined as a smooth manifold M with a smooth section g of the positive-definite quadratic forms on the tangent bundle. This means that for every point $p \in M$, the section evaluated at p yields a non-degenerate inner product g_p on the tangent space T_pM.

Since for all $p \in M$ the inner product g_p is a non-degenerate bilinear form on the finite dimensional vector space T_pM, it induces an isomorphism $\tilde{g}_p : T_pM \to T_p^*M$ that sends $v \in T_pM$ to the linear functional

$$g_p(v, \cdot).$$

Putting these together and passing to vector fields we get

$$\begin{aligned} \tilde{g} : \mathfrak{X}(M) &\to \Omega^1(M) \\ X &\mapsto (Y \mapsto g(X, Y)), \end{aligned} \tag{1.4}$$

by defining $g(X, Y)(p) = g_p(X_p, Y_p)$. This identification used Proposition 1.9.

When we think back to the initial intuition of a Morse function as a height function, it is natural to ask if there exists a vector field that 'always points upwards'. If the space is bounded, the flow lines of this vector field then start and end at critical points as we will show later.

Luckily such a vector field exists in Euclidean space, namely the gradient of a smooth function ∇f. This operator gives us at each point of $\mathbb{R}^n$ a vector pointing in the direction of the steepest slope. We wish to extend this vector field to arbitrary manifolds.

Before we define the analogue of the gradient vector field, we set up some notation and give a new perspective on some old acquaintances.

Proposition 1.10 *Let M be a smooth manifold, let $p \in M$ and let $X \in \mathfrak{X}(M)$ be a smooth vector field. Given a smooth function $f : M \to \mathbb{R}$ we can define:*

1. *$Df(X) \in C^\infty(M)$ by $Df(X)(p) = Df(p)[X(p)]$, and*
2. *$Df \in \Omega^1(M)$ by setting it to be the usual differential.*

Proof Given $p \in M$,

$$Df(X)(p) = Df(p)[X(p)] \in T_{f(p)}\mathbb{R} \cong \mathbb{R}.$$

Note that this isomorphism is canonical and thus we are really dealing with an element of $C^\infty(M)$ after this canonical identification. (All the components are smooth.)

For the second point, again use the canonical isomorphism between $\mathbb{R}$ and its tangent space at a point. The differential

$$Df(p) : T_pM \to T_{f(p)}\mathbb{R} \cong \mathbb{R}$$

can thus be seen as an element of T_p^*M and so $Df \in \Gamma(\bigwedge^1 T^*M)) = \Omega^1(M)$. Again everything is smooth by the usual argument. □

With this setup out of the way, we can move towards generalising the gradient vector fields to arbitrary manifolds. As established above, we can now make sense of the expression $\tilde{g}^{-1}(Df)$. Since $\tilde{g}$ is an isomorphism, it yields a unique vector field with the property that for all $X \in \mathfrak{X}(M)$ we have:

$$g(\tilde{g}^{-1}(Df), X) = \tilde{g}(\tilde{g}^{-1}(Df))(X) = Df(X). \tag{1.5}$$

And thus we are able to define

Definition 1.11 (Gradient vector field) *Let $f : M \to \mathbb{R}$ be a smooth function on a Riemannian manifold (M, g). Then the **gradient vector field** of f w.r.t. the metric g is the unique smooth vector field ∇f such that for all $X \in \mathfrak{X}(M)$:*

$$g(\nabla f, X) = Df(X) \in C^\infty(M).$$

Pointwise this just means that $\forall p \in M$,

$$g_p(\nabla f(p), X(p)) = Df(p)[X(p)].$$

We now take a look at the flow lines of the negative gradient vector field. Somewhat confusingly, we will refer to the 'gradient flow lines' when we really mean the negative gradient flow lines.

Definition 1.12 (Gradient flow lines) *Solutions $\gamma : \mathbb{R} \to M$ to the equation*

$$\dot{\gamma}(t) = -\nabla f(\gamma(t))$$

*are called **gradient flow lines**.*

By the local existence theorem for solutions to initial value problems, solutions to this equation exist at least locally. This means that

$$\forall x \in M \; \exists \varepsilon > 0 \text{ s.t. } \exists \gamma_x : (-\varepsilon, \varepsilon) \to M \text{ s.t. } = \begin{cases} \gamma_x(0) = x \\ \dot{\gamma}_x(t) = -\nabla f(\gamma_x(t)). \end{cases}$$

If the manifold M is compact then solutions exist for all times.

We are now able to formalise the intuition that f decreases along the gradient flow lines.

Proposition 1.13 *Let $f : M \to \mathbb{R}$ be smooth. Then f decreases along its flow lines.*

Proof

$$\begin{aligned} \frac{d}{dt} f(\gamma_x(t))) = Df(\gamma_x(t))\dot{\gamma}_x(t) &= -Df(\gamma_(t))\nabla f(\gamma_x(t)) \\ &= -Df(\nabla f)(\gamma_x(t)) \\ &= -g(\nabla f, \nabla f)(\gamma_x(t)) \\ &= -\|\nabla f\|_g^2(\gamma_x(t)) \leq 0 \end{aligned}$$

where $\|\nabla f\|_g^2 := g(\nabla f, \nabla f)$. □

If we assume that M is compact we can make a precise statement about the origin of the flow lines.

Proposition 1.14 *Let $f : M \to \mathbb{R}$ be a Morse function on a compact smooth Riemannian manifold (M, g). Then every gradient flow line of f begins and ends at a critical point, i.e. for any $x \in M$, $lim_{t\to+\infty}\gamma_x(t)$ and $lim_{t\to-\infty}\gamma_x(t)$ exist, and they are both critical points of f.*

For a proof we refer to [2, Proposition 3.19].

Another property that Morse functions enjoy is that they are dense in the space of smooth function on M. This is Theorem 5.27 in [2].

Theorem 1.15 *Let M be a compact manifold. Given any smooth function $f : M \to \mathbb{R}$ and any $\varepsilon > 0$, there is a Morse function $g : M \to \mathbb{R}$ such that* $\max\{|f(x) - g(x)| \mid x \in M\} < \varepsilon$.

1.3 Cellular homology with coefficients

In this section we quickly recap the definition of cellular homology and state the most important results. The first definition describes how one can *glue* topological spaces together.

Definition 1.16 (Attaching an n-cell) *Let Y be a topological space. We say that a topological Z is obtained from Y by attaching n-cells if there exists a pushout*

$$\begin{array}{ccc} \sqcup_{\lambda\in\Lambda} S^{n-1} & \xrightarrow{f} & Y \\ \downarrow & & \downarrow j \\ \sqcup_{\lambda\in\Lambda} B^n_\lambda & \xrightarrow{g} & Z \end{array}$$

here Λ is a (possibly uncountable) index set. The map $j : Y \to Z$ is a closed embedding, and g induces a homeomorphism $\sqcup_{\lambda\in\Lambda} E^n_\lambda \to Z \setminus Y$. We set $f_\lambda = f_{|S^{n-1}_\lambda}$ and $g_\lambda = g_{|B^n_\lambda}$. We call f_λ the attaching maps and g_λ the characteristic maps.

We then inductively build up CW-complexes.

Definition 1.17 *Let $X \subset X'$ be topological spaces such that X' is closed in X. A **cellular decomposition** of the pair (X, X') consists of a sequence of subspaces:*

$$X' = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X$$

1. *X carries the colimit or direct limit topology, i.e. a set $C \subset X$ is closed if and only if $C \cap X^n$ is closed for all $n \geq -1$.*
2. *For all $n \geq 0$, the space X^n is obtained from X^{n-1} by attaching n-cells.*

*The space X^n is called the **n-skeleton** of the cellular decomposition. If $X = X^n$ for some n, then the minimal such n is called the **dimension** of X. A **CW-complex** is a topological space that admits a cellular decomposition.*

With these definitions we are able to formulate the titular cellular homology.

Definition 1.18 (Cellular homology with coefficients) *Let X be a topological space that admits a cellular decomposition, and let R be a ring. Define a chain complex $(C^{Cell}_\bullet(X;R), \partial^{Cell})$ as follows.*

Firstly, set $C^{Cell}_n(X;R) = 0 \ \forall\ n < 0$. Given $n \geq 0$, define

$$C^{Cell}_n(X;R) = H_n(X^n, X^{n-1}; R).$$

Let $j_n : (X^n, \emptyset) \hookrightarrow (X^n, X^{n-1})$ denote the inclusion, and abbreviate by $\nu_n = H_n(j_n) : H_n(X^n;R) \to H_n(X^n, X^{n-1};R)$ the induced map on homology. For $n \geq 1$, we define the boundary operator operator $\partial^{Cell} : C^{Cell}_n(X;R) \to C^{Cell}_{n-1}(X;R)$ as the map

$$H_n(X^n, X^{n-1};R) \xrightarrow{\delta_n} H_{n-1}(X^{n-1};R) \xrightarrow{\nu_{n-1}} H_{n-1}(X^{n-1}, X^{n-2};R)$$

where δ_n is the connecting homomorphism for the long exact sequence of the pair (X^n, X^{n-1}).

The next proposition shows that we are dealing with a complex.

Proposition 1.19 *The above-defined association defines a chain complex.*

Proof All we need to show is that $(\partial^{Cell})^2 = 0$, indeed

$$H_{n+1}(X^{n+1}, X^n; R) \to H_n(X^n; R) \xrightarrow{\nu_n} H_n(X^n, X^{n-1}; R) \xrightarrow{\delta_n} H_{n-1}(X^{n-1}; R) \to H_{n-1}(X^{n-1}, X^{n-2}; R)$$

The maps ν_n and δ_n are adjacent maps in the long exact sequence of the pair (X^n, X^{n-1}) and thus their composition is 0. □

Denote $H_n^{Cell}(X; R) = H_n(C_\bullet^{Cell}(X; R), \partial^{Cell})$. The next theorem highlights the usefulness of CW-complexes.

Theorem 1.20 *Let X be a topological space that admits a cellular decomposition (X^n), and let R be a ring. Then $\forall n \geq 0$, we have*

$$H_n(X; R) \cong H_n^{Cell}(X; R).$$

Proof The proof of this is a big diagram chase and we refer to [5] for the proof. □

The following proposition shows that cellular homology gives us a tool for computing the homology of a topological space that is easier to grasp than singular homology.

Proposition 1.21 *Let X be a topological space with skeleton filtration $\emptyset \subset X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X$. Then $\forall n \geq 0$,*

$$H_k(X^n, X^{n-1}; R) = 0 \qquad \text{for } k \neq n.$$

Meanwhile $H_n(X^n, X^{n-1}; R)$ is a free R-module with basis in one-to-one correspondence to the n-cells of X.

A proof for the case $R = \mathbb{Z}$ can be found in [5].

If X is a compact space that admits a cellular decomposition we can say a bit more. Since X is compact, its dimension is finite and each X^i is obtained by attaching finitely many cells.

Corollary 1.22 *Let X be a compact CW-complex of dimension n.*

1. *If X has N_k cells of dimension k then $H_k(X; R)$ has rank at most N_k. In particular, $H_k(X; R)$ is finitely generated $\forall k$.*
2. *$H_k(X; R) = 0$ for all $k > n$.*
3. *If R is a principal ideal domain, then $H_n(X; R)$ is a free R-module.*

Proof We are only going to prove the last statement.

$$H_n^{Cell}(X;R) = \ker \partial^{Cell} : C_n^{Cell}(X;R) \to C_{n-1}^{Cell}(X;R)$$

is a submodule of $C_n^{Cell}(X;R)$. As submodules of finitely generated modules over a principal ideal domain are free, the claim follows. (A proof of this fact can for instance be found in [1].) □

1.4 CW-complex associated to a Morse function

In this section we investigate how a Morse function $f : M \to \mathbb{R}$ is able to link the different sublevel sets of f. We will show that M is homotopy equivalent to a CW-complex with k-cells in one-to-one correspondence to critical points of f with index k.

Before we go any further, let us define a smooth family of diffeomorphisms.

Definition 1.23 (one-parameter group of diffeomorphisms) *A **one-parameter group of diffeomorphisms** is a group homomorphism $\mathbb{R} \to Diff(M)$. Writing this as $t \mapsto \theta_t$, the group property tells us that*

$$\theta_0 = id, \qquad \theta_{s+t} = \theta_s \circ \theta_t \qquad \forall s,t \in \mathbb{R}. \tag{1.6}$$

We then need the following lemma.

Lemma 1.24 *A smooth vector field on M that vanishes outside of a compact set $K \subset M$ generates a unique 1-parameter group of diffeomorphisms on M.*

For a proof we relay to [3, Lemma 2.4].

Remark 1.25 *If M is compact then every vector field on M is complete, i.e. their integral curves exist for all times. Moreover, for all manifolds, there is a bijective correspondence between one-parameter subgroups of diffeomorphisms and complete vector fields.*

The next theorem shows that in between critical points, the topology (and even the smooth structure) of sublevel sets does not change.

Theorem 1.26 *Let $f : M \to \mathbb{R}$ be a smooth function on a smooth manifold. For all $a \in \mathbb{R}$, let $M^a := f^{-1}((-\infty, a]) = \{x \in M \mid f(x) \leq a\}$ be the sublevel set w.r.t. a. Let $a < b$ and assume further that $f^{-1}([a,b])$ is compact and contains no critical points of f. Then M^a is diffeomorphic to M^b, and M^a is a deformation retract of M^b.*

The intuition behind the proof is that we *drag* M^b down along the flow lines. This idea is made precise by using the above-defined one-parameter groups.

Proof (of 1.26) Let a and b be as in the theorem. Then there exists an open neighbourhood U of $f^{-1}([a,b])$ such that $\overline{U}$ is compact.

Since f has no critical points on $f^{-1}([a,b])$ the gradient vector field ∇f does not vanish on this region. Indeed, if there exists an $c \in f^{-1}([a,b])$ such that $(\nabla f)(c) = 0$, then

$$Df(c)[\nabla f(c)] = g(\nabla f, \nabla f)(c) = \|(\nabla f)(c)\|_g^2 = 0.$$

Thus c would be a critical point which is a contradiction.

Define a new vector field

$$X = \rho \nabla f$$

with $\rho : M \to \mathbb{R}$ a smooth function with compact support in U such that:

$$\rho\big|_{f^{-1}([a,b])} = \frac{1}{||\nabla f||_g^2}.$$

By the previous lemma, we get $\psi_t : M \to M$ a global one-parameter group of diffeomorphisms associated to X. For all $x \in U$ define the curve $c_x : \mathbb{R} \to \mathbb{R}$ as $c(t) = f(\psi_t(x))$. Taking the derivative at time t yields:

$$\begin{aligned}
\dot{c}_x(t) = Df(\psi_t(x)) \circ \frac{d}{dt}\psi_t(x) &= Df(\psi_t(x))(X(\psi_t(x)) \\
&= Df(X)(\psi_t(x)) \\
&= g(\nabla f, X)(\psi_t(x)) \\
&= \frac{\|\nabla f\|_g^2}{\|\nabla f\|_g^2}(\psi_t(x)) = 1.
\end{aligned}$$

Using the fundamental theorem of calculus we get

$$f(\psi_t(x)) = c_x(t) = c_x(0) + \int_0^t \dot{c}_x(t)\, dx = c_x(0) + t = f(x) + t.$$

For all $x \in f^{-1}(a)$, we have for $t = b - a$ that

$$f(\psi_{b-a}(x)) = f(x) + b - a = b.$$

Furthermore if $x \in f^{-1}((-\infty, a))$ then $f(\psi_{b-a}(x)) = f(x) + b - a < b$, this means that $\psi_{b-a}(x) \in M^b$. Hence ψ_{b-a} maps M^a diffeomorphically onto M^b.

Now for the deformation retract define

$$\begin{aligned}
R : M^b \times [0,1] &\to M^a \\
(x,t) &\mapsto \begin{cases} x & f(x) \le a \\ \psi_{t(a-(f(x))}(x) & a \le f(x) \le b. \end{cases}
\end{aligned}$$

This map is continuous by the gluing lemma. As $R(\cdot, 0) = \mathrm{id}$ and $R(\cdot, 1)$ is a retraction we have that R is a deformation retract of M^b onto M^a. □

The next theorem will do the heavy lifting for the rest of the section. It shows that critical points play a key role in the topology of sublevel sets of Morse functions. However, its proof is very technical and thus we are only going to sketch it and refer to [2] for details.

Theorem 1.27 *Let $f : M^n \to \mathbb{R}$ be a smooth function. Suppose that $a < b$, $f^{-1}([a,b])$ is compact and inside $f^{-1}([a,b])$ there is exactly one critical point. Assume that this critical point is non-degenerate and of index k. Then M^b has the homotopy type of M^a with one k-cell attached. In fact, there exists a set $e^k \subset M^b$ diffeomorphic to the closed k-disc $D^k = \{x \in \mathbb{R}^k \mid |x| \leq 1\}$ such that $M^a \cup e^k \subset M^b$ is a deformation retract of M^b.*

Proof (Sketch) Let $p \in f^{-1}([a,b])$ be the unique critical point of index k and define $c = f(p)$. By the Morse lemma there exists a chart $\varphi : U \to \mathbb{R}^n$ s.t. $\varphi(p) = 0$ and $(f \circ \varphi^{-1})(x_1, \ldots, x_n) = c - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2$.

Choose $0 < \varepsilon < 1$ small s.t. $f^{-1}([c-\varepsilon, c+\varepsilon])$ is compact and $\varphi(U)$ contains the closed ball $B_{2\varepsilon}(0)$. Consider the k-cell sitting inside U defined by

$$e^k = \varphi^{-1}(\{(x_1, \ldots x_n) \mid x_1^2 + \cdots + x_k^2 \leq \varepsilon \text{ and } x_{k+1} = \cdots = x_n = 0\}.$$

As Theorem 1.26 proves, it is enough to show that $M^{c+\varepsilon}$ has the same homotopy type as $M^{c-\varepsilon} \cup e^k$.

Define two functions to make equations more readable.

$$\xi(x_1, \ldots, x_n) = x_1^2 + \cdots + x_k^2 \quad \text{and} \quad \nu(x_1, \ldots, x_n) = x_{k+1}^2 + \cdots + x_n^2.$$

Roughly speaking these two functions give us a measure of "how much a vector lies" in the positive or negative definite plane of the tangent space. In this notation, f locally looks like

$$f = c - \xi + \nu \text{ and } e^k = \{x \in U \mid \xi \leq \varepsilon \text{ and } \nu = 0\}.$$

Now let $\mu : \mathbb{R} \to \mathbb{R}$ be any smooth function s.t. $\mu(0) \geq \varepsilon$, $\mu(r) = 0 \; \forall r \geq 2\varepsilon$ and $-1 < \mu'(r) \leq 0$. Define

$$F : M \to \mathbb{R} \text{ by } F = \begin{cases} f & \text{on } M \setminus U \\ f - \mu(\xi + 2\nu) & \text{on U.} \end{cases} \tag{1.7}$$

Then $F(p) = f(p) - \mu(0) = c - \mu(0) < c - \varepsilon$ and $F(x) \leq f(x) \; \forall x \in M$.

Here comes the part where we are going to state the results and point to the reference for details:

1. $F^{-1}((-\infty, c+\varepsilon]) = M^{c+\varepsilon}$.

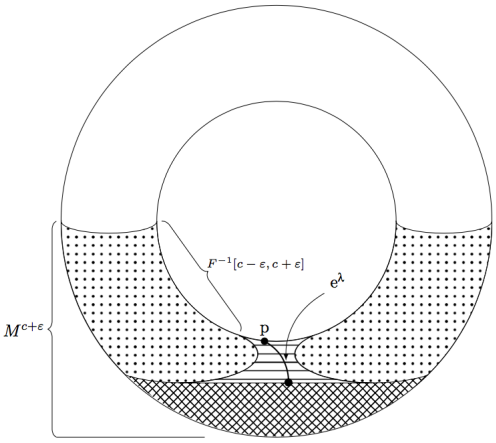


Figure 1.2: taken from [3].

2. F and f have the same critical points.
3. $F^{-1}((c-\varepsilon, c+\varepsilon]) = f^{-1}((c-\varepsilon, c+\varepsilon])$.
4. We can write $F^{-1}((-\infty, c-\varepsilon]) = M^{c-\varepsilon} \cup H$ where $H = \overline{F^{-1}((-\infty, c-\varepsilon]) \setminus M^{c-\varepsilon}}$. In addition, $M^{c-\varepsilon} \cup e^k$ is a deformation retract of $M^{c-\varepsilon} \cup H$.

Note that since $F(p) < c - \varepsilon$, the unique critical point is not in the set $F^{-1}([c-\varepsilon, c+\varepsilon])$ and hence by Theorem 1.26, $F^{-1}([-\infty, c-\varepsilon])$ is a deformation retract of $M^{c+\varepsilon}$. Combining this retract with the one from point 4 above, we get our desired deformation retract. □

Remark 1.28 *If there are j critical points $p_1, \ldots, p_j$ in the level set $f^{-1}(c)$ and not just one like in the above theorem, then if we call their respective indices $\lambda_{p_1}, \ldots, \lambda_{p_j}$ a similar proof shows that*

$$M^{c+\varepsilon} \cong M^{c-\varepsilon} \cup_{f_1} D^{\lambda_{p_1}} \cup_{f_2} \cdots \cup_{f_j} D^{\lambda_{p_j}}. \tag{1.8}$$

Where $f_1, \ldots, f_j$ are attaching maps.

Remark 1.29 *One can find a deformation retract of $F^{-1}((-\infty, c])$ to M^c. By Theorem 1.26, $F^{-1}(-\infty, c])$ is a deformation retract of $F^{-1}(-\infty, c+\varepsilon])$ and thus M^c is a deformation retract of $M^{c+\varepsilon}$.*

We now state some results from homotopy theory that will help us prove the most important theorem of this chapter. The proofs can be found in [2], Lemma 3.29 and Lemma 3.30 therein.

Lemma 1.30 (Whitehead) *Let X be a topological space and suppose that $f_0 : S^{k-1} \to X$ and $f_1 : S^{k-1} \to X$ are homotopic maps. Then there exists a homotopy equivalence*

$$h : X \cup_{f_0} D^k \to X \cup_{f_1} D^k. \tag{1.9}$$

Lemma 1.31 (Hilton) *Let X be a topological space and let $f : S^{k-1} \to X$ be an attaching map. Any homotopy equivalence $h : X \to Y$ extends to a homotopy equivalence*

$$H : X \cup_f D^k \to Y \cup_{h\circ f} D^k. \tag{1.10}$$

Theorem 1.32 (Cellular Approximation Theorem) *Every continuous map $f : X \to Y$ between cell complexes is homotopic to a cellular map, i.e. a map that takes n-skeletons to n-skeletons. If f is already cellular on a subcomplex $X' \subset X$ then the homotopy can be chosen to be constant on X'.*

We are now able to prove the main theorem of this chapter but restrict ourselves to the case where M is compact. A proof for the more general case can be found in [2].

Theorem 1.33 *Let $f : M \to \mathbb{R}$ be a Morse function on a compact smooth manifold M. Then M has the homotopy type of a CW-complex X with one cell of dimension k for each critical point of index k.*

Proof First of all we remark that M^t is compact as it is a closed set of a compact space for all $t \in \mathbb{R}$.

Now let $c_0 < c_1 < c_2 < \dots$ be the critical values of $f : M \to \mathbb{R}$. Note that this set is finite as we chose M to be compact.

The set M^t is empty for all $t < c_0$, indeed if there was a point $x \in M$ s.t. $f(x) < c_0$ then the gradient flow lines passing through x would need to end at a critical point. Hence there would exist a critical point p' s.t. $f(p') < c_0$. For $c_0 < t_0 < c_1$ the set M^{t_0} is homotopic to the discrete set of points $\{p \in \mathrm{Cr}(f) \mid f(p) = c_0\}$. This is true since all $p \in f^{-1}(c_0)$ need to have index 0 as otherwise we could flow down the manifold and repeat the previous argument to end up with a contradiction. Thus by Theorem 1.27, M^{t_0} has the same homotopy type as a collection of 0-cells, i.e. points.

We now induct on the number of critical points. Suppose that the assertion is proved for the ith critical value, i.e. for $c_i < t_i < c_{i+1}$ there exists a homotopy equivalence $h_i : M^{t_i} \to X_i$ where X_i is some CW-complex.

By Theorem 1.27 and remark 1.29 there exists an $\varepsilon > 0$ such that $M^{c_{i+1}+\varepsilon}$ has the same homotopy type as

$$M^{c_{i+1}-\varepsilon} \cup_{f_1} D^{\lambda_{p_1}} \cup_{f_2} \cdots \cup_{f_j} D^{\lambda_{p_j}}$$

with $p_l \in f^{-1}(c_{i+1})$. Moreover, by Theorem 1.26 there is a homotopy equivalence

$$g_i : M^{c_{i+1}-\varepsilon} \to M^{t_i}.$$

For every $k = 1, \dots, j$ the map $h_i \circ g_i \circ f_k : S^{\lambda_{p_k}-1} \to X_i$ is a continuous map between cell complexes and thus we can apply the cellular approximation

theorem to show that this composition is homotopic to a cellular map

$$\psi_k : S^{\lambda_{p_k}-1} \to X_i^{\lambda_{p_k}-1},$$

where $X_i^{\lambda_{p_k}-1}$ denotes the $(\lambda_{p_k}-1)$ skeleton of X_i. ($S^{\lambda_{p_k}-1}$ is the $(\lambda_{p_k}-1)$ skeleton of its own cell complex.)

Thus by lemma 1.30 and lemma 1.31 we have that

$$X_{i+1} \coloneqq X_i \cup_{\psi_1} D^{\lambda_{p_1}} \cup_{\psi_2} \cdots \cup_{\psi_j} D^{\lambda_{p_j}}$$

is a CW-complex which has the same homotopy type as $M^{c_{i+1}}$ and thus by induction this completes the proof. □

Example 1.34 *We are going to illustrate how this procedure works using the example of the vertically resting torus from Figure 1.7. This example is taken from [3]. Let f denote the height function. Theorem 1.33 describes how the topology changes when we pass the critical values $f(p), f(q), f(r)$ and $f(s)$.*

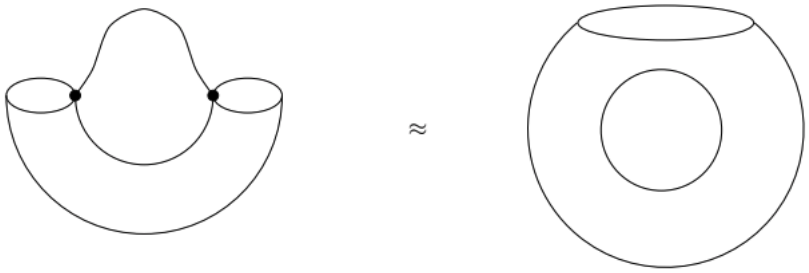
$\{-\infty\} \to f(p)$: *is the operation of attaching a 0-cell. By Theorem 1.26 we thus have that the space M^a, for $f(p) < a < f(q)$, is homotopy equivalent to a point:*



$f(p) \to f(q)$: *is the operation of attaching a 1-cell:*



$f(q) \to f(r)$: *is attaching another 1-cell:*



$f(r) \to f(s)$: *is closing up the torus by attaching a 2-cell.*

Chapter 2

Persistence Modules

In this chapter we introduce the category of persistence modules. These are central objects in topological data analysis, and in recent years there has been much progress in formalising them in the abstract language of category theory. Further down the line, we introduce the notion of an interleaving which provides a way to quantify distances between persistence modules.

2.1 Definitions

Definition 2.1 (Persistence Module) *A **persistence module** $\mathbb{V}$ is a family $(V_s)_{s\in\mathbb{R}}$ of finite-dimensional vector spaces equipped with so called inclusion morphisms $i^V_{s,t} : V_s \to V_t$ for all $s \leq t$ such that:*

1. *$\forall s \in \mathbb{R}$, $i^V_{s,s} = id$ and $\forall s \leq t \leq u$ we have that $i^V_{t,u} \circ i^V_{s,t} = i^V_{s,u}$*
2. *There exists a finite subset $F \subset \mathbb{R}$, often referred to as the spectrum of $\mathbb{V}$ such that $i^V_{s,t}$ is an isomorphism whenever s and t belong to the same connected component of $\mathbb{R} \setminus F$.*
3. *$\forall t \in \mathbb{R}$, $\varinjlim_{s>t} V_s = V_t$; equivalently for fixed t, the map $i^V_{t,s}$ is an isomorphism for $s > t$ sufficiently close to t.*

Example 2.2 *Here are a few important examples of persistence modules:*

1. *Let $[p, q) \subset \mathbb{R}$ be a non empty interval and K be a field. The **interval module** $K[p, q)$ is defined to be the persistence module with vector spaces*
$$K[p,q)_t = \begin{cases} K & \text{if } t \in [p,q) \\ 0 & \text{otherwise} \end{cases} \text{ and linear maps } i_{s,t} = \begin{cases} 1 & \text{if } s \in [p,q), t \in [p,q) \\ 0 & \text{otherwise.} \end{cases}$$
2. *Let $\mathbb{V}$ be a persistence module. Define the **shifted module** $\mathbb{V}[\delta]$ to be the persistence module with*
$$(V[\delta])_t = V_{t+\delta} \text{ and } i^{V[\delta]}_{s,t} = i^V_{s+\delta,t+\delta}.$$

Recall that given a Morse function $f : M \to \mathbb{R}$ on a compact manifold M, we may define sublevel sets $M_f^t = f^{-1}((-\infty, t])$ that form a filtration of M.

Proposition 2.3 *Let $f : M \to \mathbb{R}$ be a Morse function on a compact manifold M, and let K be a field. Then the family of spaces $(H_n(M^a; K))_{a\in\mathbb{R}}$ defines a persistence module in the sense of definition 2.1 for any n.*

Proof Singular homology $H_n(X; K)$ of any topological space X with coefficients in a field K is a K-vector space. Furthermore, each $H_n(M^a; K)$ is finite-dimensional by Proposition 1.22.

For any $s, t \in \mathbb{R}$, set $\iota_{s,t} := H_n(j_{s,t})$, where $j_{s,t} : M^s \hookrightarrow M^t$ denotes the inclusion. We now prove the individual points:

1. $\forall s \in \mathbb{R}$, $\iota_{s,s}$ is obviously the identity and $\forall s \leq t \leq u$ we have that $\iota_{t,u} \circ \iota_{s,t} = \iota_{s,u}$ as H_n is a functor.
2. Since M is compact, the set of critical values is finite, denote this set by $F = \{c_k\}_{k\in\{1,\dots,l\}}$. As seen in the proof of Theorem 1.26 M^a is a deformation retract of M^b and thus the two spaces are homotopy equivalent. Thus on the level of homology this deformation retract provides us with an isomorphism $H_n(M^a) \cong H_n(M^b)$ induced by the inclusion. Furthermore Theorem 1.27 states that the homology only changes when passing a critical value of f , and since this set F is finite we have proved the assertion.
3. Right continuity follows from Theorem 1.27 together with Remark 1.29 . These state that for each critical value c there exists an $\varepsilon > 0$ such that M^c is a deformation retract of $M^{c+\varepsilon}$, i.e. there exists an isomorphism $H_n(M^c) \cong H_n(M^{c+\varepsilon})$. □

Remark 2.4 *With this proposition it makes sense to talk of the persistence module associated to a compact manifold M together with a Morse function f on M. For each $n \in \mathbb{N}$, we will denote by $\mathcal{P}_n(f)$ the persistence module associated to $(H_n(M^a; K))_{a\in\mathbb{R}}$.*

Definition 2.5 (Morphism) *A morphism $\varphi : \mathbb{V} \to \mathbb{W}$ of persistence modules is a collection of linear maps $\varphi_s : V_s \to W_s$ such that the following commutes for all $s \leq t$:*

$$\begin{array}{ccc} V_s & \xrightarrow{i^V_{s,t}} & V_t \\ {\scriptstyle\varphi_s}\downarrow & & \downarrow{\scriptstyle\varphi_t} \\ W_s & \xrightarrow[i^W_{s,t}]{} & W_t \end{array}$$

And so we are able to speak of the category of persistence modules ***Pers****.*

Example 2.6 (δ-shift morphism) *Let $\mathbb{V}$ be a persistence module, and let $\delta > 0$. Then the* ***δ-shift morphism*** *$T^\delta : \mathbb{V} \to \mathbb{V}[\delta]$ defined by*

$$(T^\delta)_s = i^V_{s,s+\delta},$$

is a persistence morphism.

Definition 2.7 *A morphism $\varphi : \mathbb{V} \to \mathbb{W}$ is an isomorphism/injection/surjection, if each φ_t is an isomorphism/injection/surjection.*

Definition 2.8 (Image and Kernel Module) *Let $\varphi : \mathbb{V} \to \mathbb{W}$ be a persistence morphism. We define the* ***kernel persistence module*** $\ker \varphi$ *as*

$$(\ker \varphi)_t = \ker \varphi_t \;\; \textit{and} \;\; i^{\ker \varphi}_{s,t} = i^V_{s,t|\ker \varphi_s}.$$

Similarly, we define the ***image persistence module*** $\operatorname{im} \varphi$ *as*

$$(\operatorname{im} \varphi)_t = \operatorname{im} \varphi_t \;\; \textit{and} \;\; i^{\operatorname{im} \varphi}_{s,t} = i^W_{s,t|\operatorname{im} \varphi_s}.$$

It is straightforward to show that these define persistence modules.

2.2 Interleaving Distance

Definition 2.9 (ε - interleaved) *Let $\mathbb{V}$ and $\mathbb{W}$ be two persistence modules. We say that $\mathbb{V}$ and $\mathbb{W}$ are ε-interleaved if there exist morphisms $\varphi_s : V_s \to W_{s+\varepsilon}$ and $\psi_s : W_s \to V_{s+\varepsilon}$ compatible with the inclusion morphisms of the persistence modules, i.e.*

$$\psi_s \circ \varphi_{s-\varepsilon} = i^V_{s-\varepsilon,s+\varepsilon}$$
$$\varphi_s \circ \psi_{s-\varepsilon} = i^W_{s-\varepsilon,s+\varepsilon}$$

and the diagram

$$\begin{array}{ccccccc}
V_{s-\varepsilon} & \xrightarrow{\varphi_{s-\varepsilon}} & W_s & \xrightarrow{\psi_s} & V_{s+\varepsilon} & \xrightarrow{\varphi_{s+\varepsilon}} & W_{s+2\varepsilon} \\
\downarrow{\scriptstyle i^V_{s-\varepsilon,t-\varepsilon}} & & \downarrow{\scriptstyle i^W_{s,t}} & & \downarrow{\scriptstyle i^V_{s+\varepsilon,t+\varepsilon}} & & \downarrow{\scriptstyle i^W_{s+2\varepsilon,t+2\varepsilon}} \\
V_{t-\varepsilon} & \xrightarrow[\varphi_{t-\varepsilon}]{} & W_t & \xrightarrow[\psi_t]{} & V_{t+\varepsilon} & \xrightarrow[\varphi_{t+\varepsilon}]{} & W_{t+2\varepsilon}
\end{array}$$

commutes for all $s, t \in \mathbb{R}$.

ε-interleavings induce an extended pseudometric on the space of persistence modules, i.e. a pseudometric that takes values in $[0, +\infty]$. Even though it is not a distance function in the proper sense the literature refers to it as the *interleaving distance*.

Definition 2.10 (Interleaving Distance) *Let $\mathbb{V}$ and $\mathbb{U}$ be two persistence modules. Then we define*

$$d_{\text{int}} = \inf\{\varepsilon > 0 \mid \mathbb{V} \text{ and } \mathbb{U} \text{ are } \varepsilon\text{-interleaved.}\}$$

Of course merely calling it an extended pseudometric is not enough, so we are going to prove it.

Proof Let $\mathbb{U}, \mathbb{V}$ and $\mathbb{W}$ be persistence modules. We need to show the following:

1. $d_{\text{int}}(\mathbb{U}, \mathbb{U}) = 0$.
2. $d_{\text{int}}(\mathbb{U}, \mathbb{V}) = d_{\text{int}}(\mathbb{V}, \mathbb{U})$.
3. $d_{\text{int}}(\mathbb{U}, \mathbb{W}) \leq d_{\text{int}}(\mathbb{U}, \mathbb{V}) + d_{\text{int}}(\mathbb{V}, \mathbb{W})$.

1. Take $\varphi_s = i^U_{s,s}$ and $\psi_s = i^U_{s,s}$.
2. Being ε-interleaved is a symmetric property.
3. If $\mathbb{U}$ and $\mathbb{V}$ are ε_1-interleaved and $\mathbb{V}$ and $\mathbb{W}$ are ε_2-interleaved, then $\mathbb{U}$ and $\mathbb{W}$ are $(\varepsilon_1 + \varepsilon_2)$-interleaved. Indeed, let

$$\begin{aligned} \varphi^1_s &: U_s \to V_{s+\varepsilon_1}, & \varphi^2_s &: V_s \to W_{s+\varepsilon_2} \\ \psi^1_s &: V_s \to U_{s+\varepsilon_2}, & \psi^2_s &: W_s \to V_{s+\varepsilon_2} \end{aligned}$$

be the maps from the respective interleavings. Then

$$\begin{aligned} \varphi_s &= \varphi^2_{s+\varepsilon_1} \circ \varphi^1_s : U_s \to W_{s+\varepsilon_1+\varepsilon_2} \\ \psi_s &= \psi^1_{s+\varepsilon_2} \circ \psi^2_s : W_s \to V_{s+\varepsilon_1+\varepsilon_2} \end{aligned}$$

satisfy

$$\begin{aligned} \psi_s \circ \varphi_{s-\varepsilon_1-\varepsilon_2} &= \psi^1_{s+\varepsilon_2} \circ \psi^2_s \circ \varphi^2_{s+\varepsilon_2} \circ \varphi^1_{s-\varepsilon} \\ &= \psi^1_{s-\varepsilon_2} \circ i^V_{s-\varepsilon_2,s+\varepsilon_2} \circ \varphi^1_{s-\varepsilon} \\ &\overset{(\dagger)}{=} \psi^1_{s+\varepsilon_2} \circ \varphi^1_{s+\varepsilon_2-\varepsilon_1} \circ i^U_{s-\varepsilon,s+\varepsilon_2-\varepsilon_1} \\ &= i^U_{s+\varepsilon_2-\varepsilon_1,s+\varepsilon} \circ i^U_{s-\varepsilon,s+\varepsilon_2-\varepsilon_1} = i^U_{s-\varepsilon,s+\varepsilon} \end{aligned}$$

where $(\dagger)$ used the commutative diagram:

$$\begin{array}{ccc} U_{s-\varepsilon} & \xrightarrow{\varphi^1_{s-\varepsilon}} & V_{s-\varepsilon_2} \\ \downarrow{\scriptstyle i^U_{s-\varepsilon,s+\varepsilon_2-\varepsilon_1}} & & \downarrow{\scriptstyle i^V_{s-\varepsilon_2,s+\varepsilon_2}} \\ U_{s+\varepsilon_2-\varepsilon_1} & \xrightarrow[\varphi^1_{s+\varepsilon_2-\varepsilon_1}]{} & V_{s+\varepsilon_2} \end{array}$$

The proof that $\varphi_s \circ \psi_{s-\varepsilon_1-\varepsilon_2} = i^V_{s-\varepsilon,s+\varepsilon}$ goes along the same lines. Thus we have shown that $d_{\text{int}}(\mathbb{U}, \mathbb{W}) \leq \varepsilon_1 + \varepsilon_2$. Taking the infimum over ε_1 and ε_2 gives the desired result. □

Remark 2.11 *A priori, it might happen that $d_{\text{int}}(\mathbb{V}, \mathbb{W})$ vanishes for non-isomorphic $\mathbb{V}$ and $\mathbb{W}$, i.e. that the pseudometric is degenerate on the space of isomorphism classes of persistence modules. Later on when we discuss the isometry theorem, it will however become clear that we are actually dealing with an extended metric on the space of isomorphism classes of persistence modules.*

Chapter 3

Structure Theorem for Graded Modules

This chapter provides us with algebraic tools to analyse persistence modules. First, we are going to prove the Representation Theorem [11] by Zomorodian and Carlsson which roughly states that there is a one-to-one correspondence between finitely generated persistence modules over a field K and finitely generated graded modules over $K[t]$.

In the second section, we prove a classification theorem for finitely generated graded modules over a graded principal ideal. This will allow us to show that persistence modules can (up to isomorphism) be characterised by a finite set of intervals.

For the proof of the ZC-Representation Theorem, we will follow [6].

3.1 Basics about Rings and Modules

Before we start, we recall some fundamental notions of commutative algebra.

Definition 3.1 *Let R be a commutative ring.*

- *A **principal ideal** of R is an ideal of R that is generated by a single element.*
- *A **principal ideal domain (PID)** is an integral domain in which every ideal is principal.*

Definition 3.2 *Let R be a ring and let M be a left R-module. A set $B \in M$ is called a **generating set** if every element of M can be written as a finite sum of elements of B multiplied by elements of R. A subset B is called **linearly independent** if for any given distinct elements $b_1, b_2, \ldots, b_n$ of B, the only solution to*

$$r_1 b_1 + r_2 b_2 + \cdots + r_n b_n = 0_M$$

*is $r_1 = r_2 = \cdots = r_n = 0_R$. A subset B of M is called a **basis** if it is both a generating set and linearly independent. We say that M is **free** if it admits a basis. A module is called **finitely generated** (sometimes we write f.g.) if we can find a basis B such that the cardinality of B is finite.*

Definition 3.3 (Noetherian Ring) *A commutative ring R is called **Noetherian** if all ideals of R are finitely generated.*

Definition 3.4 (Noetherian Module) *Let R be a commutative ring. An R-module is called **Noetherian** if every submodule is finitely generated.*

Theorem 3.5 (Hilbert Basis Theorem) *If R is a Noetherian Ring then $R[X]$ is also Noetherian.*

Definition 3.6 (Graded Ring) *A **graded ring** R is a ring R with additive subgroups R_n, for $n \geq 0$, such that $R = \bigoplus_n R^n$, and such that $R_n \cdot R_m \subset R_{n+m}$. A **graded ring homomorphism** $f : R \to S$ between graded rings is a ring homomorphism with the property that $f(R_n) \subset S_n$ for all $n \geq 0$.*

Definition 3.7 (Graded Module) *Let R be a graded ring. A module M over R is said to be a graded module if $M = \bigoplus_n M_n$ where the M_n are abelian subgroups of M, such that $R_i M_j \subset M_{i+j}$ for all i, j. A **graded module homomorphism of degree i** between two graded modules M and N is a module homomorphism $\varphi : M \to N$ with the property that $\varphi(M_n) \subset N_{n+i}$.*

Theorem 3.8 *If M is a Noetherian R-module, N a submodule of M then the quotient module M/N is Noetherian.*

Proof Every submodule of M/N has the form L/N where L is a submodule of M with $N \subset L \subset M$. As M is Noetherian, L is f.g. and thus if $(l_1, \ldots, l_r)$ are generators of L the set $\{[l_1], \ldots, [l_r]\}$ generate L/N. □

Proposition 3.9 *Let M, M' and M'' be modules over a Noetherian ring. Then if $0 \to M' \to M \to M'' \to 0$ is exact then M is Noetherian if and only if M', M'' are.*

A proof of this can be found in [1, Proposition 6.3].

Corollary 3.10 *Let M be a module over a Noetherian ring and let $N \subset M$ be a submodule. Then the quotient module M/N is also Noetherian.*

Proof The sequence $0 \to N \to M \to M/N \to 0$ is exact. □

Lemma 3.11 *Let R be a Noetherian ring and M be an R-module. Then M is Noetherian if and only if M is finitely generated.*

Proof The module M is a submodule of M so by definition of Noetherian M is finitely generated.

Conversely, if M is a finitely generated R-module it admits a finite set of generators $(m_1, \dots, m_n)$ and we are able to construct maps

$$\begin{aligned} \varphi_i : R &\to M \\ 1_R &\mapsto m_i \quad \forall i \end{aligned}$$

Putting these together, we get a surjective map $\varphi = \bigoplus_{i=1}^{n} \varphi_i : R^n \to M$. Thus by the first isomorphism theorem for modules M is a quotient module of R^n. The graded module R^n is Noetherian by induction and any quotient module of a Noetherian module is again Noetherian by Corollary 3.10. This implies that M is Noetherian, i.e. all submodules of M are f.g. □

3.2 The ZC-Representation Theorem

A defect of persistence modules is that they need to be defined over a field. However the next result only relies on the fact that the given ring is Noetherian, and so we are going to state the next section in greater generality.

Definition 3.12 *A persistence module whose spectrum is a subset of* $\mathbb{N}$ *and for which*

$$V_i = V_{\lfloor i \rfloor}$$

is called a ***persistence module of finitely generated type****. To avoid confusion, we will call the inclusion morphisms connecting morphisms and denote them as*

$$\varphi_i := i^V_{i,i+1}.$$

Moreover, the following diagram fully characterises a persistence module of finitely generated type

$$V_0 \xrightarrow{\varphi^0} V_1 \xrightarrow{\varphi^1} V_2 \xrightarrow{\varphi^2} \dots.$$

By the definition of a persistence module there exists an $m \in \mathbb{N}$ *such that* φ_i *is an isomorphism for each* $i \geq m$ *as the spectrum of the persistence module has to be finite.*

As persistence modules are right-continuous, we can associate to each persistence module $\mathbb{V}$ a persistence module of finitely generated type:

Denote by $\{t_1, \dots, t_n\}$ the spectrum of $\mathbb{V}$. Set $t_0 = t_1 - 1$. Define

$$\chi : \mathbb{N} \to \mathbb{R} \tag{3.1}$$

$$i \mapsto \begin{cases} t_i \text{ if } i \leq n \\ t_n + 1 \text{ if } i > n \end{cases} \tag{3.2}$$

and set

$$\varphi_i := i^V_{t_i,t_{i+1}},$$

to obtain the persistence module of finitely generated type:

$$V_{\chi_0} \xrightarrow{\varphi^0} V_{\chi_1} \xrightarrow{\varphi^1} V_{\chi_2} \xrightarrow{\varphi^2} \dots.$$

One needs to be careful that going backwards might only yield a persistence module isomorphic to $\mathbb{V}$.

Theorem 3.13 (ZC Representation Theorem [11]) *Let R be a commutative ring that is Noetherian. Then we have that the category of persistence modules of finite type over R is equivalent to the category of finitely generated non-negatively graded modules over $R[t]$.*

In order to prove this theorem, we need the lay out some more groundwork.

Definition 3.14 (I-filtration) *Let R be a commutative Noetherian ring with unity. Let $I \subseteq R$ be an ideal, and V an R-module. An **I-filtration** of V is a collection $(V_n)_{n\in\mathbb{N}}$ of submodules of V such that $V = V_0 \supseteq V_1 \supseteq V_2 \supseteq \dots$ and $IV_n \subset V_{n+1}$ for all $n \in \mathbb{N}$. An I-filtration is called **I-stable** if there is an $n_0 \in \mathbb{N}$ such that $IV_n = V_{n+1}$ for all $n \geq n_0$.*

Notation: Let $I \subseteq R$ be an ideal and $(V_n)_{n\in\mathbb{N}}$ an I-filtration of a f.g. R-module V. Define $\overline{R} := \bigoplus_{i\in\mathbb{N}} I^i$ and $\overline{V} := \bigoplus_{i\in\mathbb{N}} V_i$.

We can then give an equivalent description of the fact that an I-filtration is I-stable.

Lemma 3.15 *Let R be a commutative Noetherian ring, $I \subseteq R$ an ideal, and let V be a f.g R-module, $(V_n)_{n\in\mathbb{N}}$ an I-filtration of V. Then the following are equivalent:*

1. *$\overline{V}$ is a finitely generated $\overline{R}$-module.*
2. *$(V_n)_{n\in\mathbb{N}}$ is I-stable.*

Proof 1) $\Rightarrow$ 2): As $\overline{V}$ is f.g we may choose a finite set of homogeneous generating elements $\{g_1, \dots, g_k\}$. As this set is finite, there exists one generator of maximal degree w.r.t. the other elements of the set, call this degree m. Thus every generator is contained in $\bigoplus_{j=0}^{m} V_j$. For $n \geqslant m$, each $f \in V_n$ can be written as $f = \sum a_j g_j$ where $a_j \in I^{n-\deg(g_j)}$. Recall that $IV_n \subset V_{n+1}$, this implies that $f \in I^{n-m}V_m$ and thus $V_n \subseteq I^{n-m}V_m$. So we have that $V_n = I^{n-m}V_m$.

2) $\Rightarrow$ 1): If the filtration is I-stable, then there exists a $k \in \mathbb{N}$ such that for all $n \geqslant k$, $V_n = I^{n-k}V_k$. Then $\overline{V}$ is generated by $V_0, \dots, V_k$ and those are by assumption finitely generated as they are submodules of V and R is Noetherian, this was Corollary 3.10. Thus $\overline{V}$ is finitely generated. □

With these preliminaries out of the way we can get back to the proof of the ZC-representation Theorem. The key idea in the proof is to relate back to Lemma 3.15.

Proof (theorem 3.13) Suppose we are given a persistence module of finitely generated type $\mathbb{V}$

$$V_0 \xrightarrow{\varphi^0} V_1 \xrightarrow{\varphi^1} V_2 \xrightarrow{\varphi^2} \dots.$$

such that $V_i \cong V_j$ for all $i, j \geq n$.

Equip $R[t]$ with the standard grading and define a functor α from the category of persistence modules of finitely generated type over R to graded $R[t]$-modules on objects by:

$$\alpha(\mathbb{V}) = \bigoplus_{i=0}^{\infty} V_i \tag{3.3}$$

The R-module structure is given by the sum of the structures on the individual components, and the action of t is given by

$$t \cdot (m^0, m^1 \dots) = (0, \varphi^0(m^0), \varphi^1(m^1), \dots).$$

That is, the element t shifts elements of the module up the grading.

For all i, let B_i be a finite generating set of V_i. We claim that $\cup_{i=1}^{n} B_i$ is a generating set for

$$\overline{V} = \alpha(\mathbb{V}).$$

In order to prove this, it suffices to show that every homogeneous element in $\bigoplus V_i$ is generated by $\cup_{i=1}^{n} B_i$. Fix $d \in \mathbb{N}$ and let v_d be a homogeneous element of degree d. If $d \leq n$ then v_d is generated by the elements of B_d as B_d is a generating set for V_d.

If $d > n$, define $v_n := {i^V_{n,d}}^{-1}(v_d)$. The element v_n is generated by B_n and thus v_d is generated by $i^V_{n,d}(B_d)$. By construction of the correspondence α, we have that $i^V_{n,d}(B_d) = t^{n-n}B_k$. This means that v_d is generated by B_k, which is finitely generated. As v_d was chosen arbitrarily, we have that $\alpha(\mathbb{V})$ is finitely generated.

Given a morphism $\varphi : \mathbb{V} \to \mathbb{W}$, define the functor α by:

$$\alpha(\varphi) = \bigoplus_{i=0}^{\infty} \varphi_i$$

where the $\varphi_i : V_i \to W_i$ are the morphisms between the individual vector spaces. This is obviously a functor as if $\varphi : \mathbb{V} \to \mathbb{W}$ and $\psi : \mathbb{U} \to \mathbb{V}$ are two persistence module morphisms then

$$\alpha(\varphi) \circ \alpha(\psi) = \bigoplus_{i=0}^{\infty} \varphi_i \circ \bigoplus_{i=0}^{\infty} \psi_i = \bigoplus_{i=0}^{\infty} \varphi_i \circ \psi_i = \alpha(\varphi \circ \psi)$$

and

$$\alpha(Id_{\mathbb{V}}) = \bigoplus_{i=0}^{\infty} Id_{V_i} = Id_{\alpha(\mathbb{V})}.$$

Conversely, if $\bigoplus_{i=0}^{\infty} V_i$ is a finitely generated graded module over $R[t]$, where R is a Noetherian ring, then applying the converse implication of Lemma 3.15 yields that $(V_i)_{i\in\mathbb{N}}$ is (t)-stable. Hence there exists an n such that for all $j \geq n$ we have $(t)V_j = V_{j+1}$. This allows us to construct a categorical inverse to α, namely define a persistence module of finitely generated type $\beta(\bigoplus_{i=0}^{\infty} V_i)$ by:

$$(\beta(\bigoplus_{i=0}^{\infty} V_i))_s = V_s \text{ and } \varphi_s = t \cdot \quad \text{(multiplication by } t\text{)}.$$

In addition, we claim that V_i is a finitely generated R-module for all i.

Let $B := \{b_1, \ldots, b_n\}$ be a homogeneous generating set of $\bigoplus_{i=0}^{\infty} V_i$. Set $d_j := deg(b_j)$ for $i \in \{1, \ldots, n\}$. Let n_i denote the number of elements in B with degree at most i. Define the map

$$\begin{aligned} \mu_i : R^{n_i} &\to V_i \\ e_i^{(j)} &\mapsto t^{i-d_j} b_j. \end{aligned} \tag{3.4}$$

The map μ_i then is a surjective R-linear map, which implies that V_i is a finitely generated R-module for all i. Thus $\beta(\bigoplus_{i=0}^{\infty} V_i)$ is of finite type.

On morphisms we define β as follows: Let $f : \bigoplus_{i=0}^{\infty} V_i \to \bigoplus_{i=0}^{\infty} W_i$ be a morphism between two graded $K[t]$-modules. Then

$$(\beta(f))_i = f|_{V_i} \quad \text{for all } i \in \mathbb{N}.$$

This association is evidently factorial.

Furthermore, it is easy to see that $\alpha \circ \beta$ and $\beta \circ \alpha$ equal the identity functor in their respective categories and thus we have proved that we are actually dealing with an equivalence of categories. □

3.3 Structure Theorem

In Group Theory there is a classification theorem for finitely generated abelian groups. As abelian groups are nothing other than $\mathbb{Z}$-modules, it is natural to ask if such a classification (or structure theorem) also exists for finitely generated modules.

Sadly in general the answer is no as one can see by trying to classify $\mathbb{Z}[t]$-modules. For instance $(2, t)$ is an ideal of $\mathbb{Z}[t]$ that is not a direct sum of (non-zero) cyclic $\mathbb{Z}[t]$-modules. However such a theorem exists if the module ring is a PID. Recall that given a field K, the polynomial ring $K[t]$ is a PID.

Let $M = \bigoplus_{i\in\mathbb{Z}} M_i$ be a graded module. Define $\sum^{\alpha} M$ to be a new graded module with $(\sum^{\alpha} M)_i = M_{i-\alpha}$. This notation is useful when constructing graded module homomorphisms, as we generally want to construct graded module homomorphisms of degree 0.

We define $0 \in K[t]$ to have $\deg(0) = +\infty$. Thus it makes notational sense to define $t^{+\infty} := 0$ and reformulate

$$K[t] = K[t]/(0) =: K[t]/(t^{+\infty}).$$

The following theorem is stated in [11, Theorem 2.1] and is an often cited result. This makes it all the more surprising that there seems to be no proof in the literature. The proof that we provide will be an adaptation of the proof for the structure theorem for finitely generated modules.

Theorem 3.16 (Structure Theorem for finitely generated graded modules) *Let K be a field and give $K[t]$ the standard grading. Then every finitely generated graded $K[t]$-module M is isomorphic to a direct sum of cyclic $K[t]$-modules. That is, it decomposes uniquely into the form*

$$M = \bigoplus_{i=1}^{n} \sum^{p_i} K[t]/(t^{q_i})$$

with $p_i \in \mathbb{N}$ and $q_i \in \mathbb{N} \cup \{+\infty\}$. In addition, the t^{q_j}'s are homogeneous elements in $K[t]$ such that $t^{q_j} \mid t^{q_{j+1}}$.

Proof The proof is similar to the one for the structure theorem of finitely generated non-graded modules over a PID but we need to adapt it to the special case at hand.

Let $(m_1, \ldots, m_n)$ be a homogeneous set of generators of M, such generators always exist as we can just split generators into their homogeneous components . Let $p_i := \deg(m_i)$. Define module homomorphisms

$$\begin{aligned} f_i : \sum^{p_i} K[t] &\to M \\ t^{p_i} &\mapsto m_i. \end{aligned} \tag{3.5}$$

Putting these together yields a graded $K[t]$-module homomorphisms of degree 0

$$f = \bigoplus_{i=1}^{n} f_i : \bigoplus_{i=1}^{n} \sum^{p_i} K[t] \to M.$$

Note that here we make use of the grading shift in order to let degrees match up.

Since the m_i's generate M, the map f is surjective and therefore by the first isomorphism theorem for modules we have that

$$M \cong \bigoplus_{i=1}^{n} \sum^{p_i} K[t]/\ker f.$$

By Corollary 3.10 we have that $\ker(f)$ is finitely generated. Pick homogeneous generators $k_1, \ldots, k_m$ for $\ker(f)$ and let $p'_j := \deg(k_j)$. Define a graded homomorphism

$$\begin{aligned} g_j : \sum^{p'_j} K[t] &\to \bigoplus_{i=1}^{n} \sum^{p_i} K[t] \\ t^{j'} &\mapsto k_j \end{aligned} \tag{3.6}$$

and regroup them to a graded module homomorphism

$$g = \bigoplus_{j=1}^{m} g_j : \bigoplus_{j=1}^{m} \sum^{p'_j} K[t] \to \bigoplus_{i=1}^{n} \sum^{p_i} K[t].$$

Notice that as the k_j's generate $\ker(f)$, we have that $\ker(f) = \operatorname{im}(g)$. Each k_j has a unique representation in the basis $(b_1, \ldots, b_n)$, explicitly $k_j = \sum_{i=1}^{n} a_{ij} b_i$ with $a_{ij} \in K[t]$.

Since maps between free $K[t]$-modules can be represented by matrices with entries in $K[t]$ we can define a matrix $(A)_{ij} = a_{ij}$ that represents the map g. Note that the j-th column is the coordinates of k_j.

Taking a closer look at the entries of A, we see that because both k_j and b_i are homogeneous and $k_j = \sum_{i=1}^{n} a_{ij} b_i$ we have that the a_{ij}'s also need to be homogeneous.

As $\deg(k_j) = p'_j$, this infers

$$\begin{aligned} \deg(a_{ij} b_i) = \deg(a_{ij}) + \deg(b_i) &= \deg(k_j) \\ \Rightarrow \deg(a_{ij}) &= p'_j - p_i. \end{aligned} \tag{3.7}$$

Claim 3.17 *If we have a graded homomorphism from a homogeneous basis to another homogeneous basis that is represented by a matrix with homogeneous entries, then the following operations leave the domain, the codomain basis and also the entries of the matrix homogeneous. Note that these are elementary matrix operations restricted to those that leave everything homogeneous.*

1. *swapping rows/columns.*
2. *multiplying a row/column by a unit.*
3. *multiplying a row/column by a homogeneous element and then adding/subtracting this new row/column to/from another row/column. The degree of the homogeneous element can be chosen in such a way that the resulting row/column has homogeneous entries.*

Proof (claim) 1. Swapping two rows corresponds to left-multiplication by the matrix T_{ij} where

$$T_{ij} = \begin{pmatrix} 1 & & & & \\ & 0_{(ii)} & \cdots & 1_{(ij)} & \\ & \vdots & 1 & \vdots & \\ & 1_{(ji)} & \cdots & 0_{(jj)} & \\ & & & & 1 \end{pmatrix}$$

Note that the inverse is given by $T_{ij}^{-1} = T_{ij}$ and that swapping two rows in a matrix obviously keeps the entries homogeneous.

As $R = T_{ij}T_{ij}R$ we have that in order to swap the i-th row with the j-th row, the basis of the domain also needs to swap the i-th and the j-th basis vector. This operation leaves the basis homogeneous.

Swapping two columns corresponds to right-multiplication with T_{ij}. Thus in the same way to swapping two basis elements but this time in the basis of the codomain.

2. Multiplication of a row by a unit of $K[t]$, which is an element of K, corresponds to left-multiplication by the matrix $S_i(\gamma)$ where

$$S_i(\gamma) = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \gamma & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

The inverse of this matrix is given by $S_i(\gamma)^{-1} = S_i(\gamma^{-1})$. We emphasise that γ is a unit because otherwise the matrix need not be invertible.

Thus the change of basis corresponding to multiplying the i-th row by a unit γ is multiplication of the i-th basis element in the domain by γ^{-1}. As γ^{-1} is in K, we get that the resulting basis element remains homogeneous.

The discussion is again the same if we are talking about multiplying columns by units just by replacing the words left-multiplication for right-multiplication and domain by codomain in the above argument.

3. First of all we show that it is indeed possible to multiply a row/column by a homogeneous element, add this row/column to a different row/column and preserve the homogeneity of the entries.

Recall that $\deg(a_{ij}) = p'_j - p_i$ where p'_j is the degree of the j-th basis vector of the codomain and p_i the degree of the i-th basis vector of the domain.

First we are going to analyse what the operation does on rows. Consider the i-th and the j-th row. We observe that the degree difference of elements in the same column is constant over all columns. For this consider the q-th and the p-th column. Then

$$\deg(a_{i,q}) - \deg(a_{j,q}) = p_j - p_i = \deg(a_{i,p}) - \deg(a_{j,p}). \tag{3.8}$$

Thus if we have two homogeneous elements $a_{i,q}$ and $a_{j,q}$ with $\deg(a_{j,q}) \geq \deg(a_{i,q})$ we can multiply by a homogeneous element α of $\deg(\alpha) = p_j - p_i$ such that $\deg(a_{j,q}) = \deg(\alpha \cdot a_{i,q})$.

By the above observation this assertion is independent of the column q that we chose and thus true for the whole i-th row. So multiplying the i-th row by α and then adding it to the j-th row keeps the homogeneity of the entries intact.

Secondly we realise that the above matrix operation is given by left-multiplication by the following matrix

$$R_{ji}(\alpha) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \alpha & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \vdots & 1 \end{pmatrix}$$

Where α is in the j-th row and i-th column. What left-multiplication with this matrix does is multiply the i-th row of A and than add this new row to the j-th row of A.

The inverse is given by $R_{ji}(\alpha)^{-1} = R_{ji}(-\alpha)$ and thus the basis change corresponding is given by sending the basis element b_j in the domain to $b_j - \alpha b_i$. Note that $\deg(\alpha b_i) = p_j - p_i + p_i = p_j = \deg(b_j)$ so the resulting basis stays homogeneous.

The claim for columns is again similar to the one given above. This proves the claim. □

Let us now return to the proof of the Structure Theorem. In the proof of the standard Structure Theorem for finitely generated modules over a principal ideal domain in the non graded case one proceeds to find a basis of the domain and codomain in which the matrix is diagonal.

For this apply the algorithm to get the matrix into Smith-Normal form. We then end up with two invertible matrices $S \in Mat(n \times n)$ and $T \in Mat(m \times m)$ such that SAT is diagonal.

Sadly there are steps in the algorithm that do not preserve the homogeneity of the basis an/or entries so we are going to present a slightly adapted algorithm that maintains homogeneity.

Remark 3.18 *If s and r are two homogeneous elements of $K[t]$, they are of the form $s = at^p$ and $r = bt^q$ with $a, b \in K$ and $p, q \in \mathbb{N}$. When $p \geq q$ we have that $ab^{-1}t^{p-q}r = s$ and thus $r \mid s$. So the **gcd** (Greatest common divisor) of two homogeneous elements is always the element of lowest degree.*

Adapted Smith-Normal algorithm:

Step 1: Start with column 1. If all the entries in the first column and first row are equal to 0, move on to **Step 3**. If not, look for the element of lowest degree. Swap this element to position (1,1). This operation maintains homogeneity by point 1) of the claim.

By the remark, this element is now the **gcd** of every element in that column. Thus we can perform a sequence of matrix operations as presented in claim point 3) to delete all of the other entries in that first column. These operations all maintain the homogeneity.

Step 2: If the element in position (1,1) is also the **gcd** of all the elements in the first row then we can use this element to delete all of the entries in that first row except the one in the top left corner.

If not, perform the same operation on the resulting matrix but this time on the first row. In general this will result in nonzero entries in the first column again. Now repeat Step 1 and 2.

This sequence terminates as the degree of the (1,1) entry goes down or stays the same in every step. If it stays the same going from step 1 to 2 we are done. If the degree would never stay the same we would end up with an element of degree 0 in the top left corner but an element of K divides every element of $K[t]$ so the degree will stay the same after a finite amount of steps.

Step 3: At this point we apply the algorithm inductively to the block to the right of the first column and below the first row.

We end up with a diagonal matrix with homogeneous entries. By point 2) of the claim, we can now "normalise" the entries of the matrix such that the resulting matrix will only have entries of the form t^p for $p \in \mathbb{N} \cup \{+\infty\}$.

Now we can swap the **gcd** of the diagonal to the topleft corner and again apply this step inductively to finally get a description of g in a homogeneous basis that is diagonal and of the form:

$$D = \begin{pmatrix} t^{q_1} & & \\ & t^{q_2} & \\ & & \ddots \end{pmatrix} \tag{3.9}$$

Where $t^{q_i} \mid t^{q_i+1}$.

To conclude, we have that $\mathrm{im}(g) \cong \bigoplus_i t^{q_i} \sum^{p_i} K[t]$. And thus

$$M \cong \mathrm{im}(f)/\ker(f) \cong \frac{\bigoplus_{i=1}^n \sum^{p_i} K[t]}{\mathrm{im}(g)} \cong \frac{\bigoplus_{i=1}^n \sum^{p_i} K[t]}{\bigoplus_i t^{q_i} \sum^{p_i} K[t]} \cong \bigoplus_{i=1}^n \sum^{p_i} K[t]/(t^{q_i}).$$

Uniqueness: Assume that the decomposition is not unique, i.e. there exist

$$D_1 := \bigoplus_{i=1}^n \left(\sum^{p_i^1} K[t]/(t^{q_i^1}) \right)^{m_i^1} \cong M \cong D_2 := \bigoplus_{i=1}^n \left(\sum^{p_i^2} K[t]/(t^{q_i^2}) \right)^{m_i^2} \tag{3.10}$$

with m_i^1 being the multiplicity of the pair (p_i^1, q_i^1) and m_i^2 being the multiplicity of the pair (p_i^2, q_i^2), and call the isomorphism between D_1 and D_2 f.

Define

$$I_1 = \bigcup_{i=1}^n \bigcup_{j=1}^{m_i^1} (p_i^1, q_i^1) \times \{j\} \subset \mathbb{N} \times (\mathbb{N} \cup \{+\infty\}) \times \mathbb{N}.$$

Define I_2 in the exact same way for D_2. These sets encode the multiplicities of the pairs. Then as $D_1 \neq D_2$, we have that $I_1 \setminus (I_1 \cap I_2) \neq \emptyset$.

Let $(p^1, q^1) \times \{j\} \in I_1 \setminus (I_1 \cap I_2)$. Hence there exists a copy of $\sum^{p^1} K[t]/(t^{q^1})$ in D_1 that is unmatched in D_2, and thus there exists an element $\alpha \in (D_1)_p$ such that

$$t^n \cdot \alpha = \begin{cases} 0 & \text{if } n \geq q \\ \neq 0 & \text{if } n < q \end{cases}$$

and for which

$$t^n \cdot f(\alpha) = 0 \text{ for } \begin{cases} n < q \\ n > q \text{ but not for } n = q. \end{cases}$$

If $t^n \cdot f(\alpha) = 0$ for $n < q$ then $0 = t^{q-1} \cdot f(\alpha) = f(t^{q-1} \cdot \alpha) \neq 0$.

If $t^n \cdot f(\alpha) = 0$ for $n > q$ but not for $n = q$, then $0 = f(t^q \cdot \alpha) = t^q \cdot f(\alpha) \neq 0$.

So in both cases we get a contradiction to f being a graded module homomorphism, and thus the decomposition is unique. □

3.4 Normal Form Theorem

The Structure Theorem for graded modules enables us to associate algebraic invariants to persistence modules. In this section we use it to show that persistence modules of finite type can be fully characterised by a family of intervals.

Theorem 3.19 (Normal Form Theorem) *Let $\mathbb{V}$ be a persistence module. Then there exists a finite collection $\{([p_i, q_i), m_i)\}_{i=1}^N$ of intervals $[p_i, q_i)$ with their multiplicities m_i, and with $p_i \in \mathbb{R}$ and $q_i \in \mathbb{R} \cup \{+\infty\}$, such that*

$$\mathbb{V} \cong \bigoplus_{i=1}^{n} K[p_i, q_i)^{m_i}. \tag{3.11}$$

Moreover, this data is unique up to permutations, i.e., to any persistence module there corresponds a unique such multiset.

Let us begin by stating that $\sum^{p_i} K[t]/(t^{q_i})$ with $p_i \in \mathbb{N}$ and $q_i \in \mathbb{N} \cup \{+\infty\}$ can be described as being polynomials that don't have coefficients before the p_i-th coefficient and no coefficients after the q_i-th coefficient, i.e.

$$\sum^{p_i} K[t]/(t^{q_i}) = \{a_n t^n + \cdots + a_0 \in K[t] \ : \ a_i = 0 \text{ for all } i < p_i \text{ and } i \geq q_i\}$$

Thus the j-th homogeneous part of $\sum^{p_i} K[t]/(t^{q_i})$ is given by

$$(\textstyle\sum^{p_i} K[t]/(t^{q_i}))_j = \begin{cases} K & \text{if } p_i \leq j < q_i \\ 0 & \text{else.} \end{cases} \tag{3.12}$$

Given a persistence module of finitely generated type $\mathbb{V}$ with connecting morphisms $\varphi_i : V_i \to V_{i+1}$, apply the Structure Theorem 3.16 to obtain a graded module isomorphism

$$\xi : \alpha(\mathbb{V}) \to \bigoplus_{i=1}^{n} \sum^{p_i} K[t]/(t^{q_i}). \tag{3.13}$$

Let us name the RHS D. The $K[t]$-module structure on D is given by actual multiplication by the formal variable t in the individual components.

In the proof of the ZC-representation Theorem 3.13, we constructed a map β such that $\beta(\alpha(\mathbb{V})) = \mathbb{V}$. Define $\mathbb{D} := \beta(D)$ together with connecting morphisms $\theta_i : D_i \to D_{i+1}$ that are given by multiplication by t.

As ξ is a graded module isomorphism and Theorem 3.13 is an equivalence of categories, $\beta(\xi)$ provides us with a persistence module isomorphism.

The maps $\beta(\xi)_i$ are given by simply restricting the graded module isomorphism ξ to the i-th homogeneous components, i.e. $\beta(\xi)_i = \xi_{|V_i} : V_i \to D_i$. These maps are thus K-linear isomorphisms. In addition the following commutes.

$$\begin{array}{ccccccccc} \dots & \longrightarrow & V_{i-1} & \xrightarrow{\varphi_{i-1}} & V_i & \xrightarrow{\varphi_i} & V_{i+1} & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle \beta(\xi)_{i-1}} & & \downarrow{\scriptstyle \beta(\xi)_i} & & \downarrow{\scriptstyle \beta(\xi)_{i+1}} & & \\ \dots & \longrightarrow & D_{i-1} & \xrightarrow[\theta_{i-1}]{} & D_i & \xrightarrow[\theta_i]{} & D_{i+1} & \longrightarrow & \dots \end{array} \tag{3.14}$$

We now give an explicit choice of basis on the individual K-vector spaces D_i that make it very clear what the individual connecting morphisms φ_i do.

Recall that for an arbitrary vector space W we have a canonical isomorphisms $W \cong W \bigoplus \{0\}$. Thus using equation (3.12) we have that D_i is indeed a vector space with the same dimension as V_i for all i.

An arbitrary element $f \in D$ can be written as:

$$f = (f_1, \dots, f_n) \quad \text{such that} \quad f_i \in \sum^{p_i} K[t]/(t^{q_i})$$

with $p_i \in \mathbb{N}$ and $q_i \in \mathbb{N} \cup \{+\infty\}$. With this it makes sense to write

$$(0, \dots, 0, \overbrace{t^i}^{j}, 0, \dots, 0) \in D_i$$

where $\overbrace{\ }^{j}$ is used to denote the j-th entry of the vector and by a slight abuse of notation t^i denotes the equivalence class of the polynomial $t^i \in (\sum^{p_j} K[t]/(t^{q_j}))_i$. Here we need to be careful as the j-th component of D_i might be zero again by equation (3.12) and thus t^i might be zero.

The notation makes it seem more complicated than it is so we are going to give an example to clear things up.

Example 3.20 *Let $D = \sum^1 K[t] \bigoplus \sum^2 K[t]/(t^4)$. A possible element $f \in D$ could be $f = (t + t^2, t^2 + t^3)$, i.e. $f_{|D_0} = (0,0)$, $f_{|D_1} = (t,0)$ and $f_{|D_2} = (t^2, t^2)$. The action of t is given by multiplication by t in the individual components:*

$$t \cdot f = (t^2 + t^3, t^3 + t^4) \equiv (t^2 + t^3, t^3).$$

The new notation enables us to state an explicit basis of the individual D_l's. A careful consideration of the restrictions given in (3.12) leads to:

$$\bigcup_{\substack{i=1 \\ p_i \leq l < q_i}}^{n} \{(0,\dots,0,\overbrace{t^l}^{i},0,\dots,0)\}$$

as a basis. Note that q_i might be infinite. We can pull this basis back using $(\beta(\xi)_l)^{-1}$ to get a basis of V_l.

Proposition 3.21 *Let $D = \bigoplus_{j=1}^{n} \sum^{p_j} K[t]/(t^{q_j})$. To each tuple $(p_j, q_j - p_j)$ we can associate a basis element η_j of D_{p_j} that spawns at 'time' p_j, i.e. lies in coker(θ_{p_j-1}), and dies at 'time' q_j, i.e. lies in ker($\theta_{q_j-p_j}$).*

Proof Let the connecting morphisms act on basis elements. For this assume that $(\sum^{p_j} K[t]/(t^{q_j}))_i$ is nonzero. Then

$$\theta_i(0,\dots,0,\overbrace{t^i}^{j},0,\dots,0) = \begin{cases} 0 & \text{if } i+1 = q_j \\ (0,\dots,0,\overbrace{t^{i+1}}^{j},0,\dots,0) & \text{else} \end{cases} \tag{3.15}$$

This means that the connecting morphisms send basis elements either to basis elements or to 0. Thus a basis of $\ker(\theta_i)$ can be chosen amongst basis vectors of D_i, this is something that not true for arbitrary linear map and makes this basis special.

A similar thought leads to the conclusion that $\operatorname{coker}(\theta_{i-1})$ is spanned by

$$\{(0,\dots,0,\overbrace{t^i}^{j},0,\dots,0) \mid \text{ for } j \in \{1,\dots,n\} \text{ such that } i = p_j\}. \qquad \square$$

Using the commutative diagram 3.14 we obtain a basis of each vector space V_i that has the same properties.

The next definition enables us to build persistence modules out of other ones.

Definition 3.22 (Direct Sum) *The **direct sum** $\mathbb{W} = \mathbb{U} \bigoplus \mathbb{V}$ of two persistence modules $\mathbb{U}$ and $\mathbb{V}$ is defined as:*

$$W_t = U_t \bigoplus V_t \quad , \quad w_{s,t} = u_{s,t} + v_{s,t}.$$

We can now use direct sums to either construct persistence modules from intervals or to decompose them into direct sums:

On the one hand, we can build persistence modules from an indexed family of intervals $[p_l, q_l)_{l\in\Lambda}$, i.e.

$$\mathbb{V} = \bigoplus_{l\in\Lambda} K[p_l, q_l). \tag{3.16}$$

On the other hand, given an arbitrary persistence module of finitely generated type $\mathbb{U}$, we can use the Structure Theorem to obtain

$$\alpha(\mathbb{U}) \cong \bigoplus_{i=1}^{n} \sum^{p_i} K[t]/(t^{q_i}) \tag{3.17}$$

with $p_i, q_i \in \bar{\mathbb{R}}$.

Proposition 3.21 makes it possible to identify the part isomorphic to $\sum^{p_i} K[t]/(t^{q_i})$ with $K[p_i, q_i)$. If we do this for every component in the direct sum of (3.17), we get that

$$\mathbb{U} \cong \bigoplus_{i=1}^{n} K[p_i, q_i). \tag{3.18}$$

Thus interval modules can be seen as the atomic building blocs of persistence modules.

Given any persistence modules $\mathbb{V}$ we have seen that we can associate a unique persistence module of finitely generated type, call this one $\mathbb{U}$. The decomposition of $\mathbb{U}$ induces a decomposition of $\mathbb{V}$, namely:

$$\mathbb{V} \cong \bigoplus_{i=1}^{n} K[\chi(p_i), \chi(q_i)) \tag{3.19}$$

with $p_i \in \mathbb{R}$ and $q_i \in \mathbb{R} \cup \{+\infty\}$, and χ as in equation (3.1).

This shows that any persistence module can be written as a finite sum of interval modules. Uniqueness follows from the uniqueness part of the Structure Theorem. Grouping up intervals that have the same endpoints proves Theorem 3.19.

Chapter 4

Barcodes

In the following, we are going to introduce the category of barcodes. These fundamental invariants are a direct consequence of the Normal Form Theorem 3.19 and yield us a powerful tool to classify persistence modules. Later on, we will show the space of barcodes comes equipped with a natural metric, the Bottleneck distance.

Definition 4.1 *A **barcode** B is a finite multiset of half-open intervals, i.e. it is a finite collection $\{([p_i, q_i), m_i)\}$ of intervals $[p_i, q_i)$ with given multiplicities $m_i \in \mathbb{N}$. For us, the p_i's will always belong to $\mathbb{R}$ and the q_i's will belong to $\mathbb{R} \cup \{+\infty\}$.*

Theorem 4.2 *There is a bijective correspondence between isomorphism classes of persistence modules and barcodes. We will denote the barcode associated to a persistence module $\mathbb{V}$ by $\mathcal{B}(\mathbb{V})$.*

Proof Given a persistence module $\mathbb{V}$, the Normal Form Theorem 3.19 associates a unique decomposition into interval modules to $\mathbb{V}$, i.e.

$$\mathbb{V} \cong \bigoplus_{i=1}^{n} K[p_i, q_i)^{m_i}. \tag{4.1}$$

Take the multiset $\{([p_i, q_i), m_i)\}$ to be the unique barcode associated to $\mathbb{V}$.

Conversely, given two persistence modules $\mathbb{V}_1$ and $\mathbb{V}_2$ that are isomorphic, Theorem 3.19 yields:

$$\mathbb{V}_1 \cong \bigoplus_{i=1}^{n_1} K[p_i^1, q_i^1)^{m_i^1} \text{ and } \mathbb{V}_2 \cong \bigoplus_{j=1}^{n_2} K[p_j^2, q_j^2)^{m_j^2}.$$

But this also means that

$$\mathbb{V}_1 \cong \bigoplus_{j=1}^{n_2} K[p_j^2, q_j^2)^{m_j^2}$$

by the uniqueness part of Theorem 3.19. Hence $\mathcal{B}(\mathbb{V}_1) = \mathcal{B}(\mathbb{V}_2)$. □

Remark 4.3 *Later on when we define what morphisms are in the category of barcodes, we will see that this association fails to be functorial. A counterexample will be provided in Example 4.29.*

4.1 Bottleneck Distance

In practice, the interleaving distance is hard to compute. Given the easier to grasp combinatorial nature of barcodes and the bijection of Proposition 4.2, it is natural to ask whether or not it is possible to equip the space of barcodes with an extended pseudo-metric.

To motivate the definition of such a pseudo-metric on the space of barcodes, we need to trace back to the origins of persistent homology, topological data analysis. Topological spaces and functions on them are common types of data in all disciplines of the natural sciences. Nevertheless, their complexity can be overwhelming. Thus one big objective is to identify features in data that 'stand out'.

Recall that by Remark 2.4 we are able to associate a persistence module to a Morse function.

Remark 4.4 *Given a Morse function $f : M \to \mathbb{R}$, denote by $\mathcal{B}_n(M, f)$ the barcode we get by applying Theorem 4.2 to the persistence module $\mathcal{P}_n(f)$. And define the total barcode $\mathcal{B}(M, f) := \bigcup_{n\in\mathbb{N}} \mathcal{B}_n(M, f)$.*

Barcodes are able to encode the structure of a filtration. Each interval of the form $[p, q)$ represents a 'feature' that is 'born' at p and 'dies' at q. For example the barcode $\mathcal{B}(M, f)$ can intuitively be interpreted as encoding when 'holes' open up in our filtration and when they close up again as the following example shows

The left side illustrates how the topology of the sublevel sets only changes when passing a critical point. This fact is represented on the right by the corresponding barcode.

For the identified features to be of any relevance, we need them to be stable under small perturbations of the filtration function f. In the world of topological data analysis this is called 'noise resistance'.

In the following, we define a pseudo-metric on the space of barcodes called the **bottleneck distance** for which

$$d_{\text{bot}}(\mathcal{B}_n(M, f), \mathcal{B}_n(M, g)) \leq ||f - g||_{\max} \tag{4.2}$$

holds $\forall n \in \mathbb{N}$, and where $|| \cdot ||_{\max}$ denotes the maximum norm.

This formalises our intuition that if two functions are close, then so must be their respective associated barcodes.

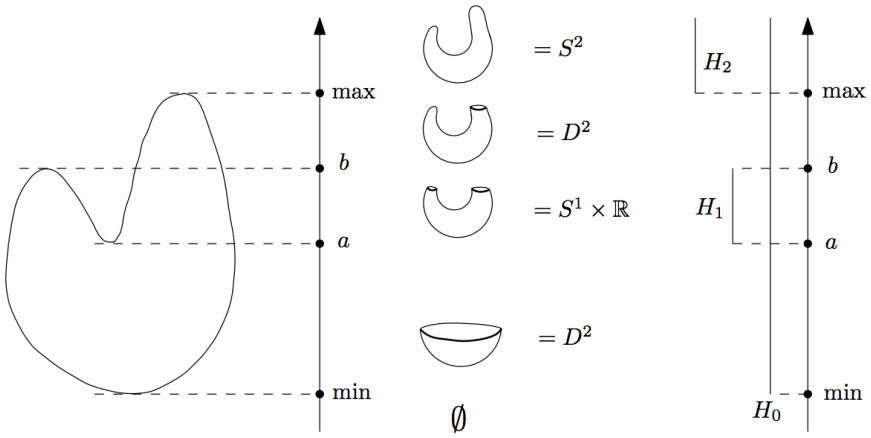


Figure 4.1: The height function on the (topological) 2-sphere and the corresponding barcode. Picture taken from [7].

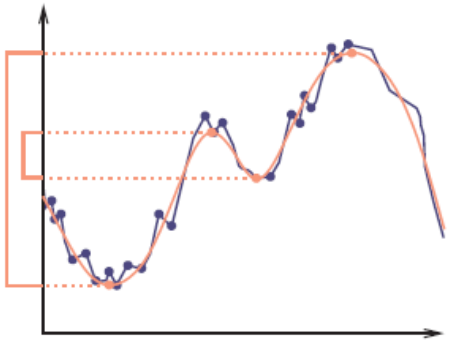


Figure 4.2: taken from [8]

In Figure 4.1, the orange function $f : \mathbb{R} \to \mathbb{R}$ represents an idealised function that we want to approximate. With real data, there are always going to be some measurement errors. Usually we are given a few data points and interpolate smoothly between them, this is represented by the blue function $g : \mathbb{R} \to \mathbb{R}$. One can see that the functions are close and thus a distance function on barcodes should quantify this.

The most important step in the formulation of such a distance function is to see that the function g has a lot more critical points than f, but these are also closer together in time than the ones of f. The emerging 'features' thus have a comparatively short lifespan to the ones of f. For example if we take the zero homology groups the 'features' would correspond to connected components emerging or closing up.

We thus want our distance function to be able to ignore barcodes $[p, q)$ for which p and q are 'close'. If we associate to $[p, q)$ the tuple (p, q) in

the extended plane $\bar{\mathbb{R}}^2$, then the barcodes with short lifespans are precisely the ones close to the diagonal $\Delta = \{(x,x) \mid x \in \mathbb{R}\}$. This motivates the following:

We extend the l^∞-norm to a distance on the extended plane by setting:

$$d^\infty((p,q),(r,s)) = \max(|p-r|,|q-s|) \quad d^\infty((-\infty,q),(-\infty,s)) = |q-s|$$
$$d^\infty((p,\infty),(r,\infty)) = |p-r| \quad d^\infty((-\infty,\infty),(-\infty,\infty)) = \infty$$
$$d^\infty((p,q),\Delta) = \frac{1}{2}(p-q).$$

One minor annoyance of barcodes is that multisets can contain multiple instances of the same interval. One way to resolve this is for instance to replace an element α with multiplicity k by $\alpha_1,\ldots,\alpha_k$. From now on we will do so implicitly.

The next definition lends us a tool to compare multisets in a sensible manner.

Definition 4.5 (Partial Matching) *A **partial matching** between two multisets A and B is a collection of pairs $M \subset A \times B$ such that:*

- *$\forall \alpha \in A$, there is at most one $\beta \in B$ such that $(\alpha,\beta) \in M$.*
- *$\forall \beta \in B$, there is at most one $\alpha \in A$ such that $(\alpha,\beta) \in M$.*

Definition 4.6 (δ-Matching) *We say that a partial matching M is a **δ-matching** if all of the following are true:*

- *if $(\alpha,\beta) \in M$ then $d^\infty(\alpha,\beta) \leq \delta$.*
- *if $\alpha \in A$ is unmatched then $d^\infty(\alpha,\Delta) \leq \delta$.*
- *if $\beta \in B$ is unmatched then $d^\infty(\beta,\Delta) \leq \delta$.*

Sometimes it will be useful to give a more categorical characterisation of a partial matching.

Proposition 4.7 *Each partial matching is fully characterised by a bijection of subsets.*

Proof Given a partial matching f between two barcodes A and B, define

$$A' := \{\alpha \in A \ : \ \exists \beta \text{ s.t. } (\alpha,\beta) \in M\}$$
$$B' := \{\beta \in A \ : \ \exists \alpha \text{ s.t. } (\alpha,\beta) \in M\}$$

Then by the definition of a partial matching, there exists a bijection of sets

$$\tilde{f} : A' \to B'.$$

Conversely, given a bijection $\tilde{f} : A' \to B'$ of subsets $A' \subset A$ and $B' \subset B$, define the partial matching

$$M := A' \times B'. \qquad \square$$

With this alternative definition of a partial matching, we are able to form the category of barcodes.

If f is a partial matching between two barcodes A and B, let $\tilde{f}$ be as in the proof of Proposition 4.7. Define

$$\mathrm{coim} f := A' \qquad \mathrm{im} f := B'$$

Let f be a partial matching between the barcodes A and B, and g be partial matching between B and C, define their composition to be the partial matching obtained from the bijection of subsets:

$$\tilde{g} \circ \tilde{f}|_{\tilde{f}^{-1}(\mathrm{im} f \cap \mathrm{coim} g)} : \tilde{f}^{-1}(\mathrm{im} f \cap \mathrm{coim} g) \to \tilde{g}(\mathrm{im} f \cap \mathrm{coim} g)$$

An example can be seen in figure 4.3

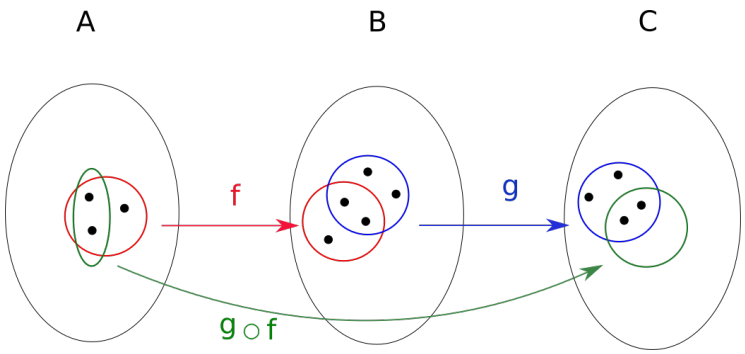


Figure 4.3: $g \circ f$ matches the two points that lie in $\mathrm{im} f \cap \mathrm{coim} g$.

It is then easy to check that this composition satisfies the axioms for a category.

Definition 4.8 *The category **Bar** has as objects barcodes, morphisms partial matchings, and composition as specified above.*

We are now able to define the bottleneck distance.

Definition 4.9 (Bottleneck Distance) *If A and B are two barcodes then*

$$d_{\mathrm{bot}}(A, B) = \inf\{\delta \mid \exists \delta\text{-matching between } A \text{ and } B\}.$$

Proposition 4.10 *The bottleneck distance is an extended metric on the space of barcodes.*

Proof Let $A = \{(I_i^a, m_i^a)\}_{i=1}^n$ and $B = \{(I_j^b, m_j^b)\}_{j=1}^l$ be two barcodes where I_i^a corresponds to the i-th bar in A and m_i^a to the multiplicity of I_i^a. We are only going to prove that

$$d_{\text{bot}}(A, B) = 0 \iff A = B$$

and leave the proof of the other points to the reader.

Assume that $d_{\text{bot}}(A, B) = 0$. This implies that we are able to construct a partial matching such that every bar in A is matched by a bar in B. Indeed, if an instance of any bar I_i^a is unmatched, it is matched with the diagonal and thus

$$d_{\text{bot}}(A, B) \geq d_\infty(I_i^a, \Delta) > 0.$$

Furthermore the matched bars need to have the same endpoints as otherwise $d_{\text{bot}}(A, B) > 0$. Thus $A = B$. □

Example 4.11 *Let us return to the previous example. Let f and g be two Morse functions as in Figure 4.1 . Associate to each function their respective barcode, explicitly*

$$B_f := \mathcal{B}(M, f) = \{[p_i^f, q_i^f) \mid i \in \{1, \ldots, n\}\}$$
$$B_g := \mathcal{B}(M, g) = \{[p_i^g, q_i^g) \mid i \in \{1, \ldots, m\}\}$$

Plot the tuples (p_i^f, q_i^f) and (p_i^g, q_i^g). Then if $d_{\text{bot}}(B_f, B_g) \leq \delta$ each tuple is either in the δ-range of a tuple of the other barcode or in the δ-range of the diagonal.

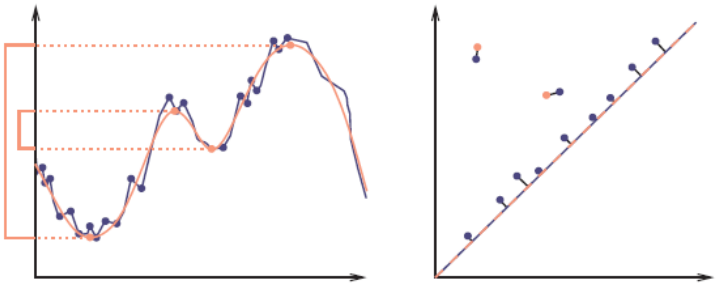


Figure 4.4: taken from [8]

With the bottleneck distance we are able to formalise what we mean by stability.

Theorem 4.12 (Stability Theorem) *Let M be a compact manifold with Morse functions $f, g : M \to \mathbb{R}$. Then the associated barcodes satisfy:*

$$d_{\text{bot}}(\mathcal{B}_n(M, f), \mathcal{B}_n(M, g)) \leq ||f - g||_{\max} \quad \forall n \in \mathbb{N} \tag{4.3}$$

where $||\cdot||_{\max}$ denotes the maximum norm.

Before we can prove the Stability Theorem, we need to establish a relationship between the interleaving distance and the bottleneck distance.

Theorem 4.13 (Isometry Theorem) *Let $\mathbb{U}$ and $\mathbb{V}$ be two persistence modules. Then*

$$d_{\text{int}}(\mathbb{U}, \mathbb{V}) = d_{\text{bot}}(\mathcal{B}(\mathbb{U}), \mathcal{B}(\mathbb{V}))$$

where $\mathcal{B}$ was defined in Theorem 4.2. The proof is deferred to section 4.2

Proposition 4.14 *Let M be a compact manifold and let $f, g : M \to \mathbb{R}$ be two Morse functions. Assume further that $||f - g||_{\max} \leq \delta$. Then the persistence modules $\mathcal{P}_n(f)$ and $\mathcal{P}_n(g)$ are δ-interleaved for all $n \in \mathbb{N}$.*

Proof Let $t \in \mathbb{R}$ and fix $\in \mathbb{N}$. If $c \in f^{-1}((-\infty, t]) = M_f^t$ then $f(c) \leq t$. For all $x \in M$ we have by assumption that

$$|g(x) - f(x)| \leq \delta$$

which implies that $g(c) < t + \delta$ and thus $c \in M_g^{t+\delta}$. From this we infer $M_f^t \subset M_g^{t+\delta}$. On the level of homology, this inclusion induces a map

$$\varphi_t : H_n(M_f^t) \to H_n(M_g^{t+\delta}).$$

Swapping the roles of f and g we get maps $\psi_t : H_n(M_g^t) \to H_n(M_f^{t+\delta})$.

Note that $(H_n(M_f^t))_{t \in \mathbb{R}}$ are precisely the vector spaces of the persistence module $\mathcal{P}_n(f)$. Also note that by functoriality of the H_n-functor, the interleaving relations are satisfied. □

We can now go back to proving the Stability Theorem.

Proof (Theorem 4.12) Let $f, g : M \to \mathbb{R}$ be Morse function such that

$$||f - g||_{\max} \leq \delta.$$

Proposistion 4.14 implies that $d_{\text{int}}(\mathcal{P}_n(f), \mathcal{P}_n(g)) \leq \delta$ from which the assertion follows from the Isometry Theorem. □

We are now also able to relate back to Remark 2.11.

Proposition 4.15 *The interleaving distance is an extended metric on the space of isomorphism classes of persistence modules.*

Proof Let $\mathbb{U}$ and $\mathbb{V}$ be two persistence modules with $d_{\text{int}}(\mathbb{U}, \mathbb{V}) = 0$. Then by the Isometry Theorem, $d_{\text{bot}}(\mathcal{B}(\mathbb{U}), \mathcal{B}(\mathbb{V})) = 0$ and thus $\mathcal{B}(\mathbb{U}) = \mathcal{B}(\mathbb{V})$ as the bottleneck distance is an extended metric by Proposistion 4.10. Theorem 4.2 then tells us that $\mathbb{V}$ and $\mathbb{W}$ are isomorphic, as they have the same barcode. □

4.2 Isometry Theorem

The isometry splits into two parts:

$$d_{\text{int}}(\mathbb{U}, \mathbb{V}) \geq d_{\text{bot}}(\mathcal{B}(\mathbb{U}), \mathcal{B}(\mathbb{V})) \tag{4.4}$$

and its converse

$$d_{\text{int}}(\mathbb{U}, \mathbb{V}) \leq d_{\text{bot}}(\mathcal{B}(\mathbb{U}), \mathcal{B}(\mathbb{V})). \tag{4.5}$$

4.2.1 Proof of $d_{\text{int}}(\mathbb{U}, \mathbb{V}) \leq d_{\text{bot}}(\mathcal{B}(\mathbb{U}), \mathcal{B}(\mathbb{V}))$

Following the treatment of the isometry theorem given in [9], we are first going to prove (4.5) for interval modules and then use Theorem 3.19 to deduce it in the general case.

Lemma 4.16 *Let $[p, q)$ be an interval, let $\mathbb{V} = K[p, q)$ be the corresponding interval module, and let 0 denote the zero persistence module. Then*

$$d_{\text{int}}(\mathbb{V}, 0) = \frac{1}{2}(q - p).$$

Proof Let $\delta \geq 0$. The persistence modules $\mathbb{V}$ and 0 are δ-interleaved if there exist

$$\varphi_t : V_t \to 0 \ , \ \psi_t : 0 \to V_{t+\delta} \quad \text{for all } t \in \mathbb{R}.$$

So the question boils down to the following question: When does $i^V_{t,t+2\delta} = 0$ for all $t \in \mathbb{R}$? A moment's thought reveals that this happens exactly when $2\delta > (q - p)$. Taking the infimum over all possible δ gives the desired result. □

The distance between two interval modules can also be given explicitly.

Lemma 4.17 *Let $[p, q)$ and $[r, s)$ be two intervals, and let $\mathbb{U} = K[p, q)$ and $\mathbb{V} = K[r, s)$. Then*

$$d_{\text{int}}(\mathbb{U}, \mathbb{V}) = d^{\infty}((p, q), (r, s)).$$

Proof Let $\delta \geq 0$. We must show that if $\delta > \max(|p - r|, |q - s|)$ then $\mathbb{U}$ and $\mathbb{V}$ are δ-interleaved. Recall that

$$U_s = \begin{cases} K & \text{if } s \in [p, q) \\ 0 & \text{else} \end{cases} \qquad V_s = \begin{cases} K & \text{if } s \in [r, s) \\ 0 & \text{else.} \end{cases}$$

Thus we see that there is not much choice for possible interleaving maps. Define:

$$\varphi_s = \begin{cases} 1_K \mapsto 1_K & \text{if } s \in [p, q) \text{ and } s + \delta \in [r, s) \\ 0 & \text{else} \end{cases}$$

$$\psi_s = \begin{cases} 1_K \mapsto 1_K & \text{if } s \in [r, s) \text{ and } s + \delta \in [p, q) \\ 0 & \text{else.} \end{cases}$$

The first step is to show that the collection of maps $\Phi = (\varphi_s)$ and $\Psi = (\psi_s)$ are persistent module morphisms. For Φ this entails verifying that

$$\begin{array}{ccc} U_t & \xrightarrow{i^U_{t,t+\varepsilon}} & U_{t+\varepsilon} \\ {\scriptstyle\varphi_t}\downarrow & & \downarrow{\scriptstyle\varphi_{t+\varepsilon}} \\ V_{t+\delta} & \xrightarrow[i^V_{t+\delta,t+\delta+\varepsilon}]{} & V_{t+\delta+\varepsilon} \end{array}$$

commutes. By the simple nature of the persistence modules involved, a moment's thought reveals that this diagram only fails to commute in two cases

1. $i^V_{t+\delta,t+\delta+\varepsilon} \circ \varphi_t = 0$ and $\varphi_{t+\varepsilon} \circ i^U_{t,t+\varepsilon} = \mathrm{id}_K$.
2. $i^V_{t+\delta,t+\delta+\varepsilon} \circ \varphi_t = \mathrm{id}_K$ and $\varphi_{t+\varepsilon} \circ i^U_{t,t+\varepsilon} = 0$.

For the first case to happen, we must have $p \leq t$ and $t + \delta \leq r$. This implies $V_{t+\delta}$ and

$$U_t = U_{t+\varepsilon} = V_{t+\delta+\varepsilon} = K.$$

But this leads to a contradiction as $\delta \leq r - t \leq r - p$ and this would infer $\delta > r - p$.

The second case leads to a contradiction in a very similar manner. Also the proof that Ψ is a persistence module morphism is analogous.

We then move on to prove that these maps actually provide an interleaving, which consists of verifying that the following diagram commutes.

$$\begin{array}{ccccc} U_t & & \xrightarrow{i^U_{t,t+2\delta}} & & U_{t+2\delta} \\ & {\scriptstyle\varphi_t}\searrow & & \nearrow{\scriptstyle\psi_{t+\delta}} & \\ & & V_{t+\delta} & & \end{array}$$

This time the only point of failure is $\psi_{t+\delta} \circ \varphi_t = 0$ and $i^U_{2\delta} = \mathrm{id}_K$, this then implies $V_{t+\delta} = 0$ and $U_t = U_{t+2\delta} = K$. The obstruction then is

$$p \leq t \text{ and } t + 2\delta < q.$$

But since $\delta > r - p$ and $\delta > q - s$ this implies that $r < p + \delta \leq t + \delta \leq q - \delta < \delta$ which is absurd. □

Proposition 4.18 *Let $\mathbb{U}_1, \mathbb{U}_2, \mathbb{V}_1, \mathbb{V}_2$ be persistence modules. Then*

$$d_{\mathrm{int}}(\mathbb{U}_1 \oplus \mathbb{U}_2, \mathbb{V}_1 \oplus \mathbb{V}_2) \leq \max(d_{\mathrm{int}}(\mathbb{U}_1, \mathbb{V}_1), d_{\mathrm{int}}(\mathbb{U}_2, \mathbb{V}_2)).$$

More generally, let $(\mathbb{U}_l \mid l \in \Lambda)$ *and* $(\mathbb{V}_l \mid l \in \Lambda)$ *be families of persistence modules indexed by the same set* Λ*, and let*

$$U_l = \bigoplus_{l \in \Lambda} \mathbb{U}_l \quad \textit{and} \quad V_l = \bigoplus_{l \in \Lambda} \mathbb{V}_l.$$

Then

$$d_{\text{int}}(\mathbb{U}, \mathbb{V}) \leq \sup_{l \in \Lambda} d_{\text{int}}(\mathbb{U}_l, \mathbb{V}_l).$$

Proof Given δ-interleavings Φ_l, Ψ_l for each pair $\mathbb{U}_l, \mathbb{V}_l$, the direct sum maps $\Phi = \bigoplus \Phi_l$, $\Psi = \bigoplus \Psi_l$ constitute a δ-interleaving of $\mathbb{U}$, $\mathbb{V}$. Thus any upper bound on the $d_{\text{int}}(\mathbb{U}_l, \mathbb{V}_l)$ is an upper bound for $d_{\text{int}}(\mathbb{U}, \mathbb{V})$. In particular, this is true for the least upper bound, or supremum. □

We are now able to prove our desired inequality.

Theorem 4.19 *Let* $\mathbb{U}$ *and* $\mathbb{V}$ *be direct sums of interval modules. Then*

$$d_{\text{int}}(\mathbb{U}, \mathbb{V}) \leq d_{\text{bot}}(\mathcal{B}(\mathbb{U}), \mathcal{B}(\mathbb{V})). \tag{4.6}$$

Proof Let $\mathbb{U} = \bigoplus_{l \in \Lambda_1} \mathbb{U}_l$ and $\mathbb{V} = \bigoplus_{l' \in \Lambda_2} \mathbb{V}_{l'}$ be decompositions into interval modules. We have to show that if there exists a δ-matching between $\mathcal{B}(\mathbb{U})$ and $\mathcal{B}(\mathbb{V})$, we have $d_{\text{int}}(\mathbb{U}, \mathbb{V}) \leq \delta$. The result then follows by taking the infimum over all possible δ.

A δ-matching between $\mathcal{B}(\mathbb{U})$ and $\mathcal{B}(\mathbb{V})$ implies the existence of a partial matching between the direct sum components of $\mathbb{U}$ and $\mathbb{V}$. Indeed, define $\Lambda = \Lambda_1 \cup \Lambda_2$. With this we can rewrite

$$\mathbb{U} = \bigoplus_{l \in \Lambda} \mathbb{U}_l \text{ and } \mathbb{V} = \bigoplus_{l \in \Lambda} \mathbb{V}_l$$

such that each pair $(\mathbb{U}_l, \mathbb{V}_l)$ is one of the following:

- a pair of δ-matched intervals, i.e. $d_{\text{bot}}(\mathcal{B}(\mathbb{U}_l), \mathcal{B}(\mathbb{V}_l)) \leq \delta$.
- The corresponding barcode $\mathcal{B}(\mathbb{U}_l)$ was unmatched, set $\mathbb{V}_l = 0$.
- The corresponding barcode $\mathcal{B}(\mathbb{V}_l)$ was unmatched, set $\mathbb{U}_l = 0$.

Thus Lemma 4.16 and 4.17 lead to $d_{\text{int}}(\mathbb{U}_l, \mathbb{V}_l) \leq \delta$ for all $l \in \Lambda$. By Proposition 4.18 we then have

$$d_{\text{int}}(\bigoplus_{l \in \Lambda} \mathbb{U}_l, \bigoplus_{l \in \Lambda} \mathbb{V}_l) \leq \sup_{l \in \Lambda}(d_{\text{int}}(\mathbb{U}_l, \mathbb{V}_l)) \leq \delta. \qquad \square$$

4.2.2 Proof of $d_{\text{int}}(\mathbb{U}, \mathbb{V}) \geq d_{\text{bot}}(\mathcal{B}(\mathbb{U}), \mathcal{B}(\mathbb{V}))$

Given a morphism between two persistence modules, the goal of this section is to construct a partial matching of their respective barcodes that connects the bottleneck and the interleaving distance in a sensible way. We start off by giving a definition that will help us compare the number of intervals in a barcode. We are going to closely follow Chapter 3 of [7].

Definition 4.20 *Let A be a barcode, and $I = [b, d)$ with $d \in \mathbb{R} \cup \{\infty\}$ an interval. Denote by $A_I^- \subset A$ the collection of all bars $[a, d) \in A$ with $a \leq b$, i.e. bars that begin no later than b and end exactly at d. (Bars are taken with multiplicity)*

Proposition 4.21 *Let $I = [b, d)$ be an interval. Assume that there exists an injective morphism $\Psi : \mathbb{V} \to \mathbb{W}$. Then $\#\mathcal{B}(\mathbb{V})_I^- \leq \#\mathcal{B}(\mathbb{W})_I^-$.*

Proof Define $E_I(\mathbb{V}) := \bigcap_{b<s<d} \operatorname{im} i_{s,d}^V \cap \bigcap_{r>d} \ker i_{d,r}^V \subset V_d$. The vector space $E_I(\mathbb{V})$ consists of precisely the elements in the filtration that do not die between b and d, and then die immediately after b. Thus $\dim E_I(\mathbb{V})) = \#\mathcal{B}(\mathbb{V})_I^-$.

As Ψ is a persistence module morphism, we have the following commutative diagram

$$\begin{array}{ccc} V_s & \xrightarrow{i_{s,t}^V} & V_t \\ {\scriptstyle \psi_s}\downarrow & & \downarrow{\scriptstyle \psi_t} \\ W_s & \xrightarrow[i_{s,t}^W]{} & W_t \end{array}$$

And thus $\psi_t(\operatorname{im} i_{s,t}^V) \subset \operatorname{im} i_{s,t}^W$ and $\psi_s(\ker i_{s,t}^V) \subset \ker i_{s,t}^W$. Utilising this, we obtain

$$\begin{aligned} \psi_d(E_I(\mathbb{V})) &= \bigcap_{b<s<d} \psi_d(\operatorname{im} i_{s,d}^V) \cap \bigcap_{r>d} \psi_d(\ker i_{d,r}^V) \\ &\subset \bigcap_{b<s<d} \operatorname{im} i_{s,d}^W \cap \bigcap_{r>d} \ker i_{d,r}^W = E_I(\mathbb{W})). \end{aligned}$$

By assumption, the linear map ψ_d is an injection and thus

$$\#\mathcal{B}(\mathbb{V})_I^- = \dim E_I(\mathbb{V})) \leq \dim E_I(\mathbb{W})) = \#\mathcal{B}(\mathbb{W})_I^-. \qquad \square$$

We can play the same game with surjections instead of injections. For this define $A_I^+ \subset A$ to be the collection of all bars of the form $[b, c)$ in A with $c \geq d$ (again accounting for multiplicity). With slight modifications to the above proof, we obtain

Proposition 4.22 *If there exists a surjection from $\mathbb{V}$ to $\mathbb{W}$, then $\#\mathcal{B}(V)_I^- \geq \#\mathcal{B}(W)_I^-$.*

The following constructions associate to a given persistence morphism a partial matching of the barcodes when we restrict to either injections or surjections.

Definition 4.23 *Suppose there exists an injection between the two persistence modules $\mathbb{V}$ and $\mathbb{W}$. Let us define the induced partial matching $\mu_{inj} : \mathcal{B}(\mathbb{V}) \to \mathcal{B}(\mathbb{W})$ as follows. For every $d \in Spec(\mathbb{V}) \cup \{+\infty\}$ sort the bars of $\mathcal{B}(\mathbb{V})$ of the form $[\cdot, d)$ in 'longest-first' order:*

$$[b_k, d) \subset \cdots \subset [b_1, d) \quad \text{in } \mathcal{B}(\mathbb{V}) \text{ with } \; b_1 \leq \cdots \leq b_k.$$

Notice that $\mathcal{B}(\mathbb{V})^-_{[b_k,d)} = \bigcup_{i=1}^k [b_i, d)$, and thus by Proposition 4.21 we have that

$$\#\mathcal{B}(\mathbb{W})^-_{[b_k,d)} \geq k.$$

Hereby it follows that there exist $c_1 \leq \cdots \leq c_l \leq b_k$ such that

$$\mathcal{B}(\mathbb{W})^-_{[b_k,d)} = \bigcup_{i=1}^{l} [c_i, d).$$

Now match the bars according to the 'longest-first' order, i.e. at each step take the longest interval from the first list and match it with the longest interval in the second list:

$$[c_l, d) \subset \cdots \subset [c_1, d) \quad \text{in } \mathcal{B}(\mathbb{W}) \text{ with } \; c_1 \leq \cdots \leq c_l.$$

Do the same for all $d \in Spec(\mathbb{V}) \cup \{+\infty\}$ to obtain a partial matching

$$\mu_{inj} : \mathcal{B}(\mathbb{V}) \to \mathcal{B}(\mathbb{W}).$$

We can then state some basic properties of this partial matching.

Proposition 4.24 *If there is an injection from $\mathbb{V}$ to $\mathbb{W}$, then the induced matching $\mu_{inj} : \mathcal{B}(\mathbb{V}) \to \mathcal{B}(\mathbb{W})$ satisfies:*

1. $\operatorname{coim} \mu_{inj} = \mathcal{B}(\mathbb{V})$,
2. *For all $[b, d) \in \mathcal{B}(\mathbb{V})$, $\mu_{inj}[b, d) = [c, d)$ with $c \leq b$.*

Proof The first assertion is simply the statement that all the bars in $\mathcal{B}(\mathbb{V})$ are matched by bars in $\mathcal{B}(\mathbb{W})$, but this readily follows from Proposition 4.21.

By the 'longest-first' matching, a bar $[b_i, d) \in \mathcal{B}(\mathbb{V})$ gets matched with a bar $[c_i, d) \in \mathcal{B}(\mathbb{W})$. Applying Proposition 4.21 again, we see that $b_i \leq c_i$ for all $1 \leq i \leq k$. □

Remark 4.25 *The induced partial matching does not depend on the particular choice of injection. Indeed all we used was the fact that* $\#\mathcal{B}(\mathbb{V})_I^- \leq \#\mathcal{B}(\mathbb{W})_I^-$ *from Proposition 4.21, but this proposition only needed the existence of any injection. The number of bars in* $\mathcal{B}(\mathbb{V})_I^-$ *and* $\mathcal{B}(\mathbb{W})_I^-$ *are independent of the chosen injection.*

If we restrict ourselves to surjections, we are able to construct an analogue.

Definition 4.26 *Suppose there exists a surjection between the two persistence modules* $\mathbb{V}$ *and* $\mathbb{W}$*. Let us define the induced partial matching* $\mu_{sur} : \mathcal{B}(\mathbb{V}) \to \mathcal{B}(\mathbb{W})$ *as follows. For every* $b \in Spec(\mathbb{V})$ *sort the bars of* $\mathcal{B}(\mathbb{V})$ *of the form* $[b, \cdot)$ *in 'longest-first' order:*

$$[b, d_k) \subset \cdots \subset [b, d_1) \quad \text{in } \mathcal{B}(\mathbb{V}) \text{ with } \; d_1 \geq \cdots \geq d_k.$$

Notice that $\mathcal{B}(\mathbb{V})^+_{[b,d_k)} = \bigcup_{i=1}^k [b, d_i)$*, and thus by Proposition 4.22 we have that*

$$\#\mathcal{B}(\mathbb{W})^+_{[b,d_k)} \geq k.$$

Hereby it follows that there exist $e_1 \geq \cdots \geq e_l \geq d_k$ *such that*

$$\mathcal{B}(\mathbb{W})^+_{[b,d_k)} = \bigcup_{i=1}^{l} [b, e_i).$$

Now match the bars according to the 'longest-first' order, i.e. at each step take the longest interval from the first list and match it with the longest interval in the second list:

$$[b, e_l) \subset \cdots \subset [b, e_1) \quad \text{in } \mathcal{B}(\mathbb{W}) \text{ with } \; e_1 \geq \cdots \geq e_l.$$

Do the same for all $d \in Spec(\mathbb{V})$ *to obtain a partial matching*

$$\mu_{sur} : \mathcal{B}(\mathbb{V}) \to \mathcal{B}(\mathbb{W}).$$

Proposition 4.27 *If there is a surjection from* $\mathbb{V}$ *to* $\mathbb{W}$*, then the induced matching* $\mu_{sur} : \mathcal{B}(\mathbb{V}) \to \mathcal{B}(\mathbb{W})$ *satisfies:*

1. $\operatorname{im} \mu_{sur} = \mathcal{B}(\mathbb{W})$,
2. *For all* $[b, d) \in \mathcal{B}(\mathbb{V})$, $\mu_{sur}[b, d) = [b, e)$ *with* $d \geq e$.

This construction is again independent of the chosen surjection. We are now able to show how a persistence module morphism induces a partial matching.

Definition 4.28 *Any persistence module morphism $\Psi : \mathbb{V} \to \mathbb{W}$ can be split as follows*

$$\mathbb{V} \xrightarrow{surjective} \operatorname{im}\Psi \xrightarrow{injective} \mathbb{W}.$$

Define the induced matching

$$\mu(\Psi) := \mu_{inj} \circ \mu_{sur} : \mathcal{B}(\mathbb{V}) \to \mathcal{B}(\mathbb{W}).$$

Unfortunately, this association **fails** to be a functor from the category of persistence modules to the one of barcodes as the following example shows:

Example 4.29 *Let I be any non-empty interval and consider the following persistence modules:*

$$\mathbb{U} = \mathbb{V} = K[I] \oplus K[I], \quad \mathbb{W} = K[I],$$

and two persistence module morphisms $\Psi : \mathbb{U} \to \mathbb{V}$ and $\Phi : \mathbb{V} \to \mathbb{W}$ given by:

$$\psi_t(a,b) = (a,0) \text{ and } \varphi_t(a,b) = b.$$

Note that $\mu(\Phi) \circ \mu(\Psi)$ matches one copy of the interval I in $\mathcal{B}(\mathbb{V})$ to the interval I in $\mathcal{B}(\mathbb{W})$. On the other hand, $\Phi \circ \Psi = 0$ and thus the induced matching is empty (no intervals get matched). And so $\mu(\Phi) \circ \mu(\Psi) \neq \mu(\Phi \circ \Psi)$.

This however can be fixed to some extent if we take the subcategory of persistence modules with morphisms restricted to either injections or surjections.

Proposition 4.30 *The association $\mu_{inj/surj}$ is a functor from the category of persistence modules with morphisms only injections/surjections to the category of barcodes.*

A proof can be found in [7, Claim 3.1.13].

In order to prove that every δ-interleaving induces a δ-matching, we need the following two lemmas. The proofs can also be found in [7] (Lemma 3.2.1 and 3.2.2).

For a barcode B, denote by B_ε the collection of bars of length $> \varepsilon$ in B.

Lemma 4.31 *Assume that the two persistence modules $\mathbb{V}$ and $\mathbb{W}$ are δ-interleaved, with interleaving morphisms $\Phi : \mathbb{V} \to \mathbb{W}[\delta]$ and $\Psi : \mathbb{W} \to \mathbb{V}[\delta]$. Consider the surjection $\Phi : \mathbb{V} \to \operatorname{im}\Phi$ and the induced matching $\mu_{sur} : \mathcal{B}(\mathbb{V}) \to \mathcal{B}(\operatorname{im}\Phi)$. Then*

1. *$\mathcal{B}(\mathbb{V})_{2\delta} \subset \operatorname{coim}\mu_{sur}$,*
2. *$\operatorname{im}\mu_{sur} = \mathcal{B}(\operatorname{im}\Phi)$, and*
3. *μ_{sur} takes $[b,d) \in \operatorname{coim}\mu_{sur}$ to $[b,d')$ with $d' \in [d-2\delta, d]$.*

Lemma 4.32 *Assume that the two persistence modules $\mathbb{V}$ and $\mathbb{W}$ are δ-interleaved, with interleaving morphisms $\Phi : \mathbb{V} \to \mathbb{W}[\delta]$ and $\Psi : \mathbb{W} \to \mathbb{V}[\delta]$. Consider the injection $\operatorname{im}\Phi \to \mathbb{W}[\delta]$ and the induced matching $\mu_{inj} : \mathcal{B}(\operatorname{im}\Phi) \to \mathcal{B}(\mathbb{W}[\delta])$. Then*

1. *$\mathcal{B}(\operatorname{im}\Phi) = \operatorname{coim}\mu_{inj}$,*
2. *$\mathcal{B}(\mathbb{W}[\delta])_{2\delta} \subset \operatorname{im}\mu_{inj}$, and*
3. *μ_{inj} takes $[b,d) \in \operatorname{coim}\mu_{inj}$ to $[b',d)$ with $b' \in [b-2\delta, b]$.*

Theorem 4.33 *Let $\mathbb{V}$ and $\mathbb{W}$ be two persistence modules that are δ-interleaved. Then this interleaving induces a δ-matching between their respective barcodes $\mathcal{B}(\mathbb{V})$ and $\mathcal{B}(\mathbb{W})$.*

Before we go on to prove this, we show how the claim of this section follows from this Theorem.

Corollary 4.34 *Let $\mathbb{V}$ and $\mathbb{W}$ be two persistence modules, then*

$$d_{\text{int}}(\mathbb{V},\mathbb{W}) \geq d_{\text{bot}}(\mathcal{B}(\mathbb{V}),\mathcal{B}(\mathbb{W})).$$

Proof Let $d_{\text{int}}(\mathbb{V},\mathbb{W}) = \delta$. Then for all $\delta' > \delta$, we have that there exists a δ'-interleaving between $\mathbb{V}$ and $\mathbb{W}$. By Theorem 4.33 this induces a δ'-matching between $\mathcal{B}(\mathbb{V})$ and $\mathcal{B}(\mathbb{W})$, and thus $d_{\text{bot}}(\mathcal{B}(\mathbb{V}),\mathcal{B}(\mathbb{W})) < \delta'$. Taking the infimum over all δ' yields the result. □

Proof (of Theorem 4.33) Let $\mathbb{V}$ and $\mathbb{W}$ be two δ-interleaved persistence modules, and denote by $\Phi : \mathbb{V} \to \mathbb{W}[\delta]$ one of the respective interleaving morphisms. This persistence morphism induces a partial matching

$$\mu(\Phi) : \mathcal{B}(\mathbb{V}) \to \mathcal{B}(\mathbb{W}[\delta]).$$

Recall the map $T^\delta : \mathcal{B}(\mathbb{W}[\delta]) \to \mathcal{B}(\mathbb{W})$ from Example 2.6 that shifts all bars by δ to the right, i.e. $[a,b) \to [a+\delta, b+\delta)$. The claim is then that $T^\delta \circ \mu(\Phi)$ is a δ-matching between $\mathcal{B}(\mathbb{V})$ and $\mathcal{B}(\mathbb{W})$. By Lemma 4.31 and Lemma 4.32 we have the following diagram

$$\mathcal{B}(\mathbb{V})_{2\delta} \xrightarrow{\mu_{\text{sur}}} \mathcal{B}(\operatorname{im}\Phi) \xrightarrow{\mu_{\text{inj}}} \operatorname{im}\mu_{\text{inj}} \xrightarrow{T^\delta} \mathcal{B}(\mathbb{W}).$$

Again by Lemma 4.31 and Lemma 4.32 (the third points), the induced partial matching $\mu(\Phi)$ matches a bar $[a,b) \in \mathcal{B}(\mathbb{V})_{2\delta}$ with a bar $[a',b') \in \mathcal{B}(\mathbb{W}[\delta])_{2\delta} \subset \operatorname{im}\mu_{\text{inj}}$, where $b-2\delta \leq b' \leq b$ and $a-2\delta \leq a' \leq a$. Finally T^δ then maps $[a',b')$ to $[a'+\delta, b'+\delta)$.

Note that $\mathcal{B}(\mathbb{V})_{2\delta} \subset \operatorname{coim}\mu_{\text{sur}}$ and thus every interval in $\mathcal{B}(\mathbb{V})_{2\delta}$ is matched with an interval in $\mathcal{B}(\operatorname{im}\Phi)$. Furthermore, $\mathcal{B}(\operatorname{im}\Phi) = \operatorname{coim}\mu_{\text{inj}}$. Combining the two yields that every interval in $\mathcal{B}(\mathbb{V})_{2\delta}$ is indeed matched with an interval in $\mathcal{B}(\mathbb{W}[\delta])_{2\delta}$. Similarly any bar in $\mathcal{B}(\mathbb{W})_{2\delta}$ is matched.

Finally, a moment's thought reveals that $|a - (a' + \delta))| \leq \delta$ and $|b - (b' + \delta))| \leq \delta$. Thus we conclude that $T^\delta \circ \mu(\Phi)$ provides a δ-matching between $\mathcal{B}(\mathbb{V})$ and $\mathcal{B}(\mathbb{W})$. □

4.3 Completion of the space of barcodes

The space of barcodes is not complete with respect to the bottleneck distance, as one can see from the following example. Consider the sequence of barcodes $B_n := \bigcup_{i=1}^{n} [0, \frac{1}{n})$. This sequence is a Cauchy sequence as for $m \geq n$

$$d_{\text{bot}}(B_n, B_m) \leq \max_{n+1 \leq i \leq m} d_\infty((0, \frac{1}{i}), \Delta) = \frac{1}{2(n+1)}$$

since one possible partial matching is to match all the bars in B_n with their respective counterpart in B_m and then match the remaining bars in B_m with the diagonal.

Claim 4.35 *The limit w.r.t. the bottleneck distance of the above described sequence needs to consist of infinitely many bars.*

Proof Assume by way of contradiction that there exists a finite barcode B_∞ such that

$$\lim_{n \to \infty} B_n = B_\infty.$$

W.l.o.g we can assume that B_∞ is of the form:

$$B_\infty = \bigcup_{i=1}^{k} [0, q_i).$$

Let $q := \min_i q_i$, choose $\varepsilon > 0$ such that $q > 2\varepsilon > 0$, then as B_∞ is the limit there exists an $M \in \mathbb{N}$ such that for all $m \geq M$

$$d_{\text{bot}}(B_m, B_\infty) < \varepsilon.$$

By the definition of a partial matching, the bar $[0, q)$ can only be matched with bars $[0, \frac{1}{i})$ for which $|\frac{1}{i} - q| < \varepsilon$. But this shows that we can always choose $m' \geq M$ large enough such that $B_{m'}$ contains a bar $[0, \frac{1}{i'})$ too small to be matched with $[0, q)$. On the other hand, this bar cannot be matched by any other bar in B_∞ as q was chosen to be minimal. Thus this bar needs to be matched with the diagonal and thus

$$d_{\text{bot}}(B_{m'}, B_\infty) \geq d_\infty((0, \frac{1}{i'}), \Delta) = \frac{1}{2i'}.$$

Thus for $\varepsilon' < \frac{1}{2i'}$ we cannot find an $M' \in \mathbb{N}$ such that $d_{\text{bot}}(B_l, B_\infty) < \varepsilon'$ for all $l \geq M'$. This contradicts B_∞ being the limit. □

It is however easy to give an explicit definition of the completion of the space of barcodes.

Proposition 4.36 *The completion of the space of barcodes $\mathcal{B}$ with respect to the bottleneck distance consists of the set of infinite barcodes which satisfy the following finiteness property: for any $\varepsilon > 0$, only finitely many intervals of the barcode have length greater than ε. Denote this space by $\bar{\mathcal{B}}$.*

This is Proposition 20 in [10].

As $\bar{\mathcal{B}}$ is the completion of $\mathcal{B}$ w.r.t. d_{bot}, we can extend the bottleneck distance to $\bar{\mathcal{B}}$. Let $A, B \in \bar{\mathcal{B}}$ and let $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ denote Cauchy sequences of barcodes that have A and B as limits. (These exist as $\mathcal{B}$ is dense in $\bar{\mathcal{B}}$.) Then

$$d_{\text{bot}}(A, B) := \lim_{n\to\infty} d_{\text{bot}}(A_n, B_n).$$

To show that this definition is well defined is a standard analysis exercise.

Recall that in remark 4.4 we associated barcodes to Morse functions. The Stability Theorem 4.12 makes it possible to extend this association to smooth functions. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of Morse functions w.r.t. the maximum norm. This means that

$$\forall \varepsilon > 0 \ , \ \exists N \in \mathbb{N} \ , \ \forall n, n' \geq N \ : \ ||f_n - f_{n'}||_{\max} < \varepsilon.$$

For the same ε and N as above, the Stability Theorem then implies that

$$\forall n, n' \geq N \ , \ d_{\text{bot}}(\mathcal{B}(M, f_n), \mathcal{B}(M, f_{n'})) \leq ||f_n - f_{n'}||_{\max} < \varepsilon.$$

This means that $(\mathcal{B}(M, f_n))_{n\in\mathbb{N}}$ is a Cauchy sequence w.r.t. the Bottleneck distance. Recall that Morse functions are dense in the space of smooth functions by Theorem 1.15. So let $f \in C^\infty(M)$, then there exists a Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ of Morse functions that converges to f in the maximum norm, i.e. $||f - f_n||_{\max} \to 0$. Define

$$\mathcal{B}(M, f) := \lim_{n\to\infty} \mathcal{B}(M, f_n).$$

Note that this limit is well-defined. Let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ be two Cauchy sequences of Morse functions that converge to the same smooth function h. Then

$$d_{\text{bot}}(\lim_{n\to\infty} \mathcal{B}(M, f_n), \lim_{n\to\infty} \mathcal{B}(M, g_n)) \overset{\text{def}}{=} \lim_{n\to\infty} d_{\text{bot}}(\mathcal{B}(M, f_n), \mathcal{B}(M, g_n)) \leq \lim_{n\to\infty} ||f_n - g_n||_{\max}.$$

But this goes to 0 as both f_n and g_n converge to h in the maximum norm.

Example 4.37 *We can use this extension to associate a barcode to the constant function*

$$\begin{aligned} f : [-1, 1] &\to \mathbb{R} \\ x &\mapsto 0. \end{aligned}$$

The sequence of smooth functions $(f_n)_{n\in\mathbb{N}}$ *with*

$$\begin{aligned} f_n : [-1,1] &\to \mathbb{R} \\ x &\mapsto \frac{1}{n}(1-x^2) \end{aligned}$$

is obviously a sequence of Morse functions s.t. $\lim_{n\to\infty} ||f - f_n||_{\max} = 0$. *Furthermore,* $\mathcal{B}([-1,1], f_n) = \{[0, \frac{1}{n})\}$ *and thus*

$$\mathcal{B}([-1,1], f) \overset{def}{=} \lim_{n\to\infty} \mathcal{B}([-1,1], f_n) = \varnothing,$$

the empty barcode.

Chapter 5

Applications

5.1 Approximating functions on S^1

In this section we are going to loosely follow example 4.2.7 in [7] but we adapt it to the case of S^1.

Definition 5.1 (Boundary depth) *Let A be a barcode. The length of the longest finite interval in A is called the **boundary depth** of A, and it is denoted by $\beta(A)$. If a barcode consists only of infinite bars, we set β to zero.*

Theorem 5.2 *For a barcode A sort the lengths of finite bars in decreasing order:*

$$\beta_1 \geq \beta_2 \geq \cdots \geq \beta_N.$$

For all $\beta_i > N$ define $\beta_i = 0$. Then the function β_k is 2-Lipschitz on the space of barcodes for all $k \in \mathbb{N}$.

Proof Let A and B be two barcodes. We want to show that if $d_{\text{bot}}(A, B) = \delta$, then for all $\delta' > \delta$ we have that

$$|\beta_k(A) - \beta_k(B)| \leq 2\delta' \quad \text{for all } k \in \mathbb{N}. \tag{5.1}$$

The claim then follows as usual by taking the infimum over all δ'. In order to show this, fix a δ'-matching and assume w.l.o.g that $\beta_k(A) \geq \beta_k(B)$. If $\beta_k(A) \leq 2\delta'$, inequality (5.1) trivially holds, and thus we can further assume that

$$\beta_k(A) > 2\delta'.$$

The δ'-matching yields by definition a partial matching where some intervals of length less than $2\delta'$ get discarded and the remaining intervals are matched such that the length difference in each couple is less than $2\delta'$. Denote the lenghts of the matched intervals in decreasing order

$$a_1 \geq \cdots \geq a_N,$$
$$b_1 \geq \cdots \geq b_N.$$

A standard result from optimal transport states that

Lemma 5.3 (Matching Lemma) *For any two sets of points in* $\mathbb{R}$, $a_1 \geq \cdots \geq a_N$ *and* $b_1 \geq \cdots \geq b_N$, *we have*

$$\min_{\sigma \in S_N} \max_i |a_i - b_{\sigma(i)}| = \max_i |a_i - b_i|.$$

In other words, the 'optimal' matching is the monotone one.

This is Lemma 4.1.1 in [7].

Our δ'-matching can be represented by $\sigma' \in S_N$, namely the interval with length a_i is matched with $b_{\sigma'(i)}$. But then the matching lemma tells us that

$$\max_i |a_i - b_i| \leq \max_i |a_i - b_{\sigma'(i)}| \leq 2\delta' \implies |a_i - b_i| \leq 2\delta' \text{ for all } i.$$

Since we assumed that $\beta_k(A) > 2\delta'$, it could not have been discarded in the partial matching and thus $a_k = \beta_k(A)$. On the other hand $b_k \leq \beta_k(B)$ as some bar longer than b_k might have been discarded. Thus we conclude:

$$\beta_k(A) - \beta_k(B) \leq a_k - b_k \leq 2\delta'. \qquad \square$$

Consider a Morse function $f : S^1 \to \mathbb{R}$, where f is the height function of a 'deformed hot dog'. In the following, we are going to investigate how well we can approximate f by another Morse function g with exactly two critical points. To be more precise, we are interested in bounding $||f - g||_{\max}$ from below.

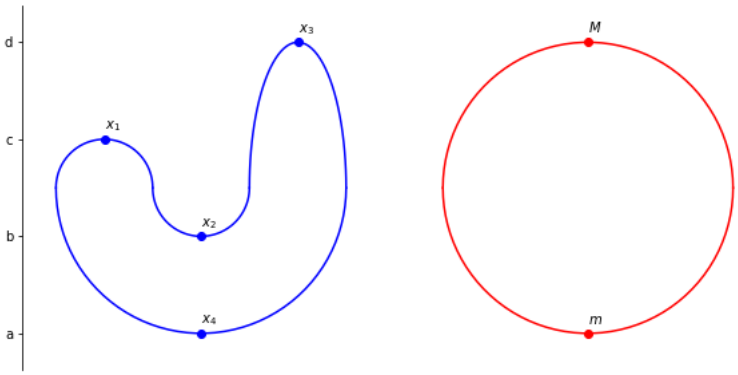


Figure 5.1: Height function of deformed hot dog on the left and circle on the right. The axis on the left denotes the critical values.

We can formulate the following table for the homology with coefficients in a field K for the sublevel sets of f:

$f^{-1}((\infty, t))$	Homotopic to	H_0	H_1	H_i for $i \geq 2$
$t < a$	$\emptyset$	0	0	0
$a < t < b$	$\{x_1\}$	K	0	0
$b < t < c$	$\{x_1\} \sqcup \{x_2\}$	K$\oplus$K	0	0
$c < t < d$	$\{x_1\}$	K	0	0
$d < t$	S^1	K	K	0

The degree 0 barcode is thus equal to $\mathcal{B}_0(f) = \{[a, +\infty), [b, c)\}$. The finite interval $[b, c)$ corresponds to the fact that for $b < t < c$, the sublevel set $f^{-1}((-\infty, t))$ has two connected components that merge for $t > c$.

The degree 1 barcode is simpler to interpret. It takes the form $\mathcal{B}_1(f) = \{[d, +\infty)\}$. This barcode encodes that as we pass d, we complete a circle.

The barcode of f thus looks like:

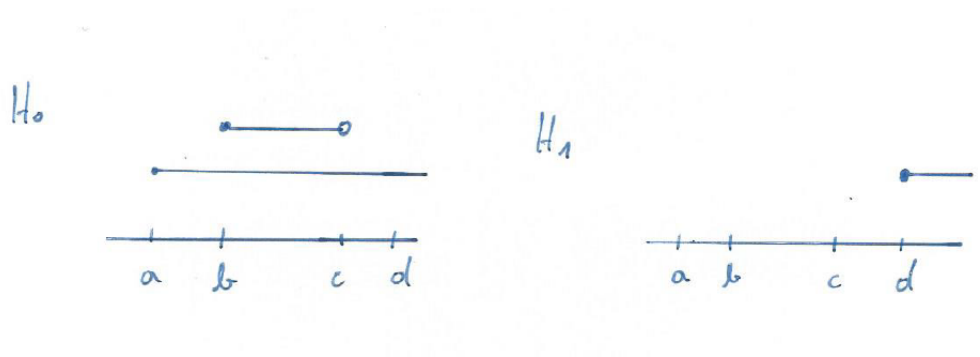


Figure 5.2: $\mathcal{B}(f)$

A Morse function $g : S^1 \to \mathbb{R}$ that has the same minimum and maximum as f but that has exactly two critical points consists of only two infinite bars,

$$\mathcal{B}_0(g) = \{[a, +\infty)\} \text{ and } \mathcal{B}_1(g) = \{[d, +\infty)\}.$$

By definition, we have that the boundary depth $\beta_1(\mathcal{B}(f)) = c - b$ and $\beta_1(\mathcal{B}(g)) = 0$ (as there are no finite intervals).

This enables us to solve the approximation question:

$$\beta_1(\mathcal{B}(f)) - \beta_1(\mathcal{B}(g)) \overset{\text{Thm 5.2}}{\leq} 2d_{\text{bot}}(\mathcal{B}(f), \mathcal{B}(g)) \overset{\text{Stability Thm}}{\leq} 2||f - g||_{\max}.$$

Hence the Stability Theorem provides us with an obstruction to approximating f, explicitly

$$||f - g||_{\max} \geq \frac{1}{2}(c - b).$$

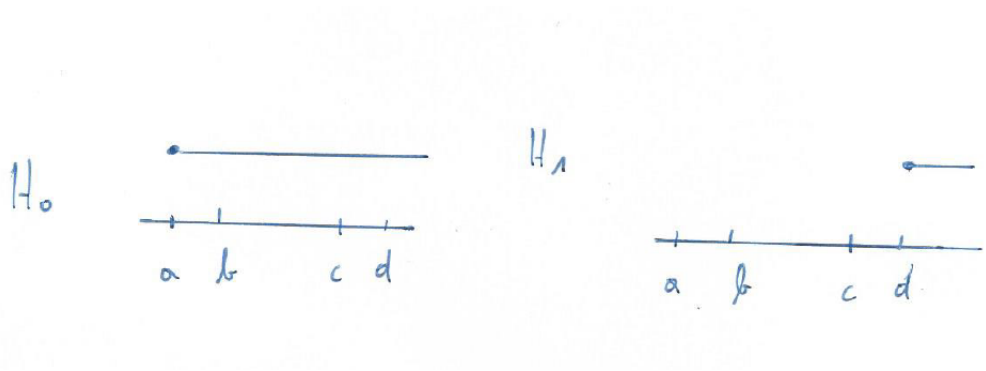


Figure 5.3: $\mathcal{B}(g)$

5.2 Topological function theory

5.2.1 The fundamental inequality

Denote by $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z}^2)$ the 2-Torus equipped with the euclidean metric $dx_1^2 + dx_2^2$, and let $\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2}$ denote the Laplace-Beltrami operator. We will follow Chapter 6 in [7] very closely.

In this final application, we provide a bound on the 'size' of a barcode induced by a Morse function $f : \mathbb{T}^2 \to \mathbb{R}$. Define the following invariant:

$$l(f) := \text{length}(\mathcal{B}(f) \cap [\min f, \max f]). \tag{5.2}$$

It measures the total length of all finite bars of f and of the segments of infinite rays in the interval $[\min f, \max f]$. The main result is Theorem 6.4.1 in [7]:

Theorem 5.4 *For every Morse function $f : \mathbb{T}^2 \to \mathbb{R}$ we have:*

$$l(f) \leq 3(||f||_2 + ||\Delta f||_2), \tag{5.3}$$

with $||\cdot||_2$ being the L_2-norm.

It will take us a while to prove this theorem. For a surface with boundary A, denote by $\zeta(A)$ for the total Betti number of A and let $|\partial A|$ denote the number of boundary components.

Proposition 5.5 *For every 2-dimensional submanifold $A \subset \mathbb{T}^2$ with non-empty boundary, the following holds*

$$\zeta(A) \leq |\partial A| + 2.$$

The proof is basically just an application of the classification theorem of 2-dimensional compact connected surfaces with boundary. We refer to Proposition 6.4.5 in [7] for it.

Definition 5.6 (Banach Indicatrix) *For a Morse function $f : \mathbb{T}^2 \to \mathbb{R}$, denote by $u(t)$ the number of connected components of the level set $f^{-1}(t)$. Define the* ***Banach Indicatrix*** *as*

$$I := \int_{\min f}^{\max f} u(t)dt.$$

For regular values t, we have $u(t) = |\partial M^t|$. We then have the fundamental inequality

$$l(f) = \int_{\min f}^{\max f} \zeta(M^t)dt \overset{\text{Prop. 5.5}}{\leq} \int_{\min f}^{\max f} (|\partial M^t| + 2)dt \leq 3\int_{\min f}^{\max f} u(t)dt = 3I. \tag{5.4}$$

5.2.2 Normal lifts of the level lines

Identify the unit circle bundle over the torus

$$U\mathbb{T}^2 := \{(x, \xi) \in T\mathbb{T}^2 \mid \xi \in T_x\mathbb{T}^2 \ , \ |\xi| = 1\}$$

with the 3-torus $\mathbb{T}^2 \times S^1$. Equip it with the *Sasaki metric* ρ given by $dx_1^2 + dx_2^2 + d\varphi^2$, where (x_1, x_2) are the Euclidean coordinates on $\mathbb{T}^2$ and φ is the polar angle of the tangent vector ξ.

Let γ be a connected component of the regular level set $f^{-1}(t)$. Let

$$\nu = \frac{\nabla f}{|\nabla f|}$$

be the vector field of unit normals along γ. Choose the parametrisation $\gamma(t)$ of γ by the Riemannian length such that $|\gamma'(t)| = 1$ for all t. Define

$$\tilde{\gamma}(t) = (\gamma(t), \nu(t)) \in U\mathbb{T}^2.$$

This is the lift of γ to $U\mathbb{T}^2$.

To prove the theorem, we need the following two lemmas.

Lemma 5.7 *The length of the tangent vector to $\tilde{\gamma}$ w.r.t. the Sasaki metric is given by*

$$|\tilde{\gamma}'|_\rho^2 = 1 + \frac{< Hess_f \gamma', \gamma' >^2}{|\nabla f|^2}. \tag{5.5}$$

For a proof we refer to [7].

Lemma 5.8 *The curve $\tilde{\gamma}$ is not contractible in $U\mathbb{T}^2$.*

Proof The statement is obvious if γ is non-contractible in $\mathbb{T}^2$. If γ is contractible in $\mathbb{T}^2$, then the curve $\tilde{\gamma}$ is homotopic to the fibre S^1 of the fibre bundle $U\mathbb{T}^2 \to \mathbb{T}^2$. Let us examine the Puppe sequence of the unit circle bundle:

$$\pi_2(\mathbb{T}^2) \to \pi_1(S^1) \cong \mathbb{Z} \to \pi_1(U\mathbb{T}^2).$$

Now $\pi_2(\mathbb{T}^2) = 0$, and thus S^1 is not contractible in $U\mathbb{T}^2$. □

We can now go back to prove the main theorem.

Proof (Theorem 5.4) Recall that $u(t)$ is the number of connected components of $f^{-1}(t)$. For a regular value t of f, decompose the level sets $f^{-1}(t)$ into connected components

$$f^{-1}(t) = \gamma_1 \cup \cdots \cup \gamma_{u(t)}.$$

Set

$$L(t) = \sum_{i=1}^{u(t)} |\tilde{\gamma}_i|_\rho^2.$$

Then as by lemma 5.8 any simple closed curve on $\mathbb{T}^2$ is non-contractible in $U\mathbb{T}^2$, we have that the ρ-length of each $\tilde{\gamma}_i$ is $\geq 2\pi$, and thus

$$L(t) \geq 2\pi u(t).$$

It follows that

$$I = \int_{\min f}^{\max f} u(t)dt \leq \frac{1}{2\pi} \int_{\min f}^{\max f} L(t)dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt \int_{f^{-1}(t)} \sum_{i=1}^{u(t)} |\tilde{\gamma}_i|_\rho^2 ds$$

Thus by lemma 5.7, we have

$$I \leq \int_{-\infty}^{+\infty} dt \int_{f^{-1}(t)} \left(1 + \frac{||Hess_f||_{op}^2}{|\nabla f|^2}\right)^{\frac{1}{2}} ds =: J,$$

where $||\cdot||_{op}$ stands for the operator norm. As f is Lipschitz, we can apply the co-area formula to obtain

$$J = \frac{1}{2\pi} \int_{\mathbb{T}^2} (|\nabla f|^2 + ||Hess_f||_{op}^2)^{\frac{1}{2}} d\mu \leq \frac{1}{2\pi} \left(\int_{\mathbb{T}^2} (|\nabla f|^2 + ||Hess_f||_{op}^2) d\mu \right)^{\frac{1}{2}}$$

where $d\mu = dx_1 dx_2$ stands for the euclidean area and for the second inequality we used Cauchy-Schwartz.

In [7] the authors show that $\int_{\mathbb{T}^2} ||Hess_f||_{op}^2 d\mu \leq ||\Delta f||_2$. In addition,

$$\left| \int_{\mathbb{T}^2} |\nabla f|^2 d\mu \right| = \left| \int_{\mathbb{T}^2} f \Delta f d\mu \right| \leq ||f||_2 ||\Delta f||_2.$$

Putting these together yields the final result

$$I \leq (\int_{\mathbb{T}^2} |\nabla f|^2 + ||Hess_f||_{op}^2 d\mu)^{\frac{1}{2}} \leq (||\Delta f||_2(||f||_2 + ||\Delta f||_2))^{\frac{1}{2}} \leq ||f||_2 + ||\Delta f||_2.$$

This result together with 5.4 proves the theorem. □

Example 5.9 *Let $\mathcal{T}_\lambda$ be the space of trigonometric polynomials on $\mathbb{T}^2$ spanned by $\sin(n_1x_1 + n_2x_2)$ and $\cos(n_1x_1 + n_2x_2)$ with $(n_1^2 + n_2^2) \leq \lambda$. It is easy to see that for $p \in \mathcal{T}_\lambda$ we have that*

$$\Delta p = -(x_1^2 + x_2^2)p.$$

And thus

$$||\Delta p||_2 \leq \lambda ||p||_2.$$

We can then apply Theorem 5.4 to obtain

$$l(p) \leq 3(1 + \lambda)||p||_2. \tag{5.6}$$

If we take for example $p(x) = \sin(nx_1) + \sin(nx_2)$, one can see that p has n^2 homologically visible critical points, i.e. maxima or minima. Furthermore, the polynomial p has $2n^2$ saddle points. The barcode of p consists of infinite rays $[2, +\infty)$, $[-2, +\infty)$ with multiplicity 2, and $[0, +\infty)$ with multiplicity 1. There are also some finite bars $[-2, 0)$ and $[0, 2)$ of multiplicty $n^2 - 1$ each. It thus follows that

$$l(p) = 4n^2 + 4.$$

We have that $\lambda = n^2$ and $||p||_2 = 2\pi$ and thus (5.6) does indeed hold.

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