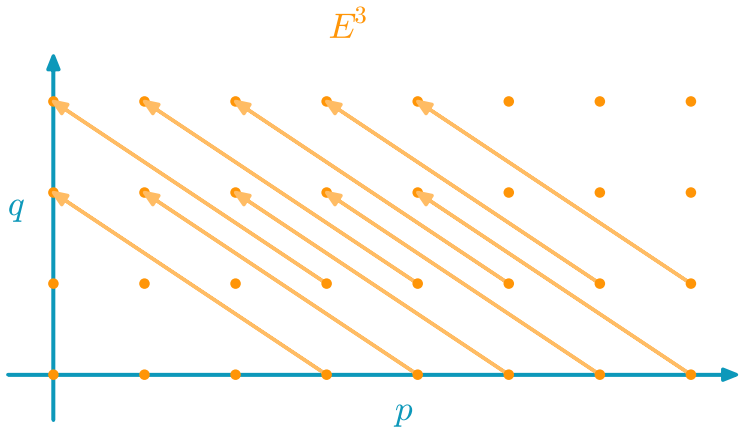
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Spectral Sequences

A Geometric Interpretation of the Serre Spectral Sequence

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Introduction

Spectral sequences are one of the major computational tools in Algebraic Topology. They allow us to compute (co)homology by an iterative process that approximates the (co)homology groups in an infinite procedure. In some sense, spectral sequences are a generalisation of long exact sequences, as they are able to link very particular long exact sequences that are interconnected.

Initially introduced by Jean Leray in 1946 [1] in his work on sheaf cohomology, the abstract algebraic formalism of spectral sequences was used by Jean-Pierre Serre to investigate an inherently geometric object, nowadays called a Serre fibration. Let E and B be topological spaces, recall that a Serre fibration is a continuous map $p : E \to B$ that satisfies the homotopy lifting property (see Definition 2.1) with respect to any CW-complex. If B is path-connected, then all the fibres $F_b := p^{-1}(b)$ are weakly equivalent and so we will just denote them by F in the following.

In his seminal Ph.D. thesis [2], Serre was able to link the (co)homology of the total space E, the base space B and the fibre F with the aid of a spectral sequence that is now named after him, the Serre spectral sequence. This thesis is devoted to exploring this spectral sequence in its many facets, and to showing why one should care about spectral sequences.

Let us now present Serre's main motivation for developing his theory. Let (B, b_0) be a pointed space. The loop space of (B, b_0) is the pointed space $\Omega(B, b_0) := \{u : I \to B \mid u(0) = u(1) = b_0\}$. The loop space construction can be iterated, and we have the following important result for the homotopy groups of a topological space B:

$$\pi_n(B) \cong \pi_{n-k}(\Omega^k B) \qquad \forall 1 \leq k \leq n.$$

Thus loop spaces are paramount to the computation of higher homotopy groups. However, if $B \neq \{\text{pt}\}$, most of our tools for computing the (co)homology of ΩB fail, as then ΩB does not have the homotopy type of a finite cell complex.

This is where the Serre spectral sequence comes to the rescue. Define $PB := \{u : I \to B \mid u(0) = b_0\}$ and the map $p : PB \to B$ by $p(u) = u(1)$. Then p is a fibration with fibre ΩB, and thus sometimes we can use the Serre spectral sequence to compute the (co)homology of ΩB.

In addition, as we will see later on, fibre bundles are Serre fibrations. For a trivial fibre bundle $F \to F \times B \to B$, we already know how to link all the different

spaces involved in the fibration. Namely with the Künneth Theorem that tells us that we have a short exact sequence:

$$0 \to \bigoplus_{i+j=k} H_i(X) \otimes H_j(Y) \to H_k(X \times Y) \to \bigoplus_{i+j=k-1} \mathrm{Tor}(H_i(X), H_j(Y)) \to 0.$$

If we take the Serre spectral sequence of a trivial fibration, we recover the Künneth formula. Thus in a way, the Serre spectral sequence can also be seen as a way of extending the Künneth formula to non trivial fibre bundles.

Outline

The first chapter exists to lay the theoretical foundation for spectral sequences and introduce a lot of the formalism. First, we define what a spectral sequence is in the language of abstract algebra and show how they arise in the context of two different constructions, once from filtrations and once from exact couples.

We proceed by introducing topological objects called fibrations, and show the necessity of extending the coefficient systems of singular homology in order to encode the highly non-trivial relation between the fibres and the total space. We show how fibrations induce a coefficients system that will serve as the natural setting in which we describe the Serre spectral sequence.

We conclude the chapter by using our newly-built formalism to pick some low-hanging fruit.

The second chapter focuses on the work of Alexandru Oancea [9] who was able to construct the Serre spectral sequence purely in the language of Morse cohomology. First of all, we recall the construction of cellular (co)homology but with the added touch of a local coefficient system. We go on to show the equivalence between singular and cellular (co)homology, and use it to give a description of the Serre spectral sequence in terms of cellular (co)homology. We analyse the trajectory space of a gradient vector field associated to a Morse function in order to construct the Thom-Smale-Witten complex of a compact manifold.

To round off this thesis, we show how a Morse function on the base space of a fibration induces the construction of a Morse function of the total space, that carries all the topological data of the fibration. This carefully constructed Morse function gives rise to the Serre spectral sequence in the context of Morse cohomology.

Acknowledgement

I would like to thank my advisor Prof. Will Merry for supervising the present work. His office door (Zoom meeting) was always open whenever I ran into problems. I would particularly like to thank him for introducing me to the Morse theoretic part of this thesis, as it turned out to be a great source of inspiration.

Chapter I

The Serre Spectral Sequence

Chapter I is broken into three parts. In the first part, we are going to set the algebraic foundations for what is to come. We are going to give a proper definition of what a spectral sequence is and show how spectral sequences naturally arise from filtrations.

In the second part, we first lay out some topological groundwork and then move on to stating and proving the existence of the Serre spectral sequence. We finish the chapter by showing why spectral sequences are such an important tool in Algebraic Topology through some carefully chosen examples.

1 Algebraic Side

This section is going to provide us with the algebraic tools necessary for what is to come. Although at times the algebra may seem like an endless battle against super- and subscripts, going through this hardship is unavoidable if one wants to tame the giant bookkeeping-beast that is spectral sequences. In the following exposition, we are going to closely follow [3, Ch. 2].

1.1 Definitions and Basic Properties

Definition 1.1. A **differential bigraded module** over a ring R is a collection of R-modules, $\{E_{p,q}\}$ where p and q are integers, together with an R-linear mapping $d : E_{\bullet,\bullet} \to E_{\bullet,\bullet}$, the **differential** of bidegree:

- $(-r, r-1)$ in the **homological** case,
- $(r, 1-r)$ in the **cohomological** case,

for some integer r. The map d should in addition satisfy $d \circ d = 0$.

Remark 1.2. In the statements and proofs, we will mostly concern ourselves with the homological case. The cohomogical case often functions completely analogously and it often boils down to flipping minus signs or reversing inclusions. In the cohomological case we will flip sub- and superscripts, e.g. we will write $E_r^{p,q}$ for the differential bigraded modules. We will also adapt this convention for all other future constructions.

With the differentials defined, we can take the homology of a differential bigraded module:

$$H_{p,q}(E_{\bullet,\bullet}, d) := (\ker d : E_{p,q} \to E_{p-r,q+r-1})/(\operatorname{im} d : E_{p+r,q-1+r} \to E_{p,q})$$

Combining the two leads us to the definition of a spectral sequence.

DEFINITION 1.3. A **spectral sequence** is a collection of differential bigraded R-modules $E^r_{\bullet,\bullet}$, where $r = 1, 2, \ldots$. The differentials are either all:

- of bidegree $(-r, r-1)$ (homological type),
- or of bidegree $(r, 1-r)$ (cohomological type),

such that for all p, q and r the module $E^{r+1}_{p,q}$ is isomorphic to $H_{p,q}(E^r_{\bullet,\bullet}, d^r)$.

It is worth noting that in order to define a spectral sequence, we need information about all the differentials. More precisely, the differential d^r defines $E^{r+1}_{\bullet,\bullet}$, but information about the differential d^{r+1} is not included in this data.

We call the r-th stage the E^r-term or the E^r-page. This terminology is in analogy to a book with infinitely many pages, where each application of the homology functor turns the page. The customary picture is the following: for $r = 1, 2, \ldots$ the r-th page is a lattice with the lattice-points consisting of the R-modules $E^r_{p,q}$ with $(p, q) \in \mathbb{Z}^2$. The arrows are the differentials d^r.

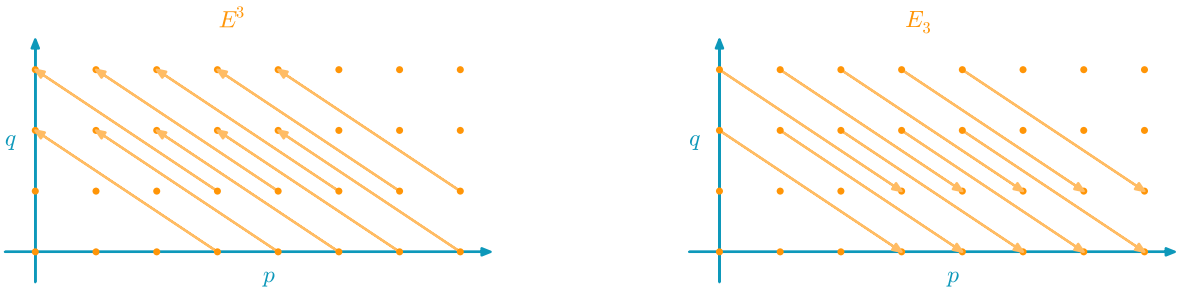


Figure I.1: Customary homological picture on the left, and cohomological picture on the right. Not all differentials were drawn for the sake of clarity.

In order to motivate this construction, suppose we are given a graded R-module $H_\bullet$ that we want to compute. This could for instance be the total (co)homology of a topological space. Suppose further that $H_\bullet$ is filtered, i.e.

$$\begin{cases} \{0\} \subset F_0 H_\bullet \subset \cdots \subset F_{n-1} H_\bullet \subset F_n H_\bullet \subset \cdots \subset H_\bullet & \text{(homological filtration)}, \\ \{0\} \subset \cdots \subset F_n H_\bullet \subset F_{n-1} H_\bullet \subset \cdots \subset F_0 H_\bullet \subset H_\bullet & \text{(cohomological filtration)}, \end{cases}$$

and assume that each $F_iH_\bullet$ is also a graded module. From this data, define the **associated bigraded R-module** by

$$E^0_{p,q}(H_\bullet, F_\bullet) := \begin{cases} F_pH_{p+q}/F_{p-1}H_{p+q} \text{ , in the homological case,} \\ F_pH_{p+q}/F_{p+1}H_{p+q} \text{ , in the cohomological case.} \end{cases}$$

Under certain conditions, it will be possible to reconstruct the graded module $H_\bullet$ from its associated bigraded module. If we can identify $E^0_{p,q}(F_\bullet H_\bullet)$ we will be one step closer to finding $H_\bullet$.

DEFINITION 1.4. A spectral sequence $\{E^r, d^r\}$ is said to be **first quadrant** if $E^r_{p,q} = 0$ whenever $p < 0$ or $q < 0$.

LEMMA 1.5. *For fixed $(p, q) \in \mathbb{Z}^2$, the R-modules $E^r_{p,q}$ stabilises. By this we mean that there exists $N \in \mathbb{N}$ such that if $r \geq N$ we have that $E^r_{p,q} \cong E^{r+1}_{p,q}$. We call this page the limit page $E^\infty_{p,q} \cong E^r_{p,q}$ for $r \geq N$.*

Proof. Set $N := \max(p + 1, q + 2)$. The assertion then follows by observing that

$$\ker(d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}) = E^r_{p,q} \qquad \text{since } p - r < 0,$$

and that

$$\operatorname{im}(d^r_{p+r,q-r+1} : E^r_{p+r,q-r+1} \to E^r_{p,q}) = \{0\} \qquad \text{since } q - r + 1 < 0.$$

□

Thus we see that in the above case, after turning the page often enough, all subsequent pages look the same (at least in fixed bidegree and up to isomorphism). This inspires us to look for a generalisation of the limit page to arbitrary spectral sequences.

For the sake of readability, we will temporarily drop the subscripts in what follows. Denote by

$$Z^2 := \ker d^2 \text{ and } B^2 := \operatorname{im} d^2.$$

The condition, $d^2 \circ d^2 = 0$ implies $B^2 \subset Z^2 \subset E^2$. By definition $E^3 \cong Z^2/B^2$. Write $\bar{Z}^3 := \ker d^3$. Since $\bar{Z}^3$ is a submodule of E^3, it can be written as Z^3/B^2, where Z^3 is a submodule of Z^2. Similarly $\bar{B}^3 := \operatorname{im} d^3$ is isomorphic to B^3/B^2 and so by the Third Isomorphism Theorem we have

$$E^4 \cong \bar{Z}^3/\bar{B}^3 \cong (Z^3/B^2)/(B^3/B^2) \cong Z^3/B^3.$$

We are led to the following tower of inclusions:

$$B^2 \subset B^3 \subset Z^3 \subset Z^2 \subset E^2.$$

Iterating this process, we present the spectral sequence as an infinite tower of submodules of E^2:

$$B^2 \subset B^3 \subset \cdots \subset B^n \subset \cdots \subset Z^n \subset \cdots \subset Z^3 \subset Z^2 \subset E^2$$

with the property that $E^{n+1} \cong Z^n/B^n$.

The differential d^{n+1} can thus be seen as a map $Z^n/B^n \to Z^n/B^n$ which has kernel Z^{n+1}/B^n and image B^{n+1}/B^n. This gives rise to a short exact sequence

$$0 \longrightarrow Z^{n+1}/B^n \longrightarrow Z^n/B^n \xrightarrow{d^{n+1}} B^{n+1}/B^n \longrightarrow 0 \tag{I.1}$$

from which we can read off the isomorphism $Z^n/Z^{n+1} \cong B^{n+1}/B^n$ for all $n \in \mathbb{N}$.

Definition 1.6. We say that an element that lies in Z^r **survives to the r-th stage** if it has been in the kernel of the previous $r-2$ differentials. The submodule B^r of E^2 is the set of all elements that are **boundaries by the r-th stage**.

Let $Z^\infty = \bigcap_n Z^n$ be the submodule of E^2 of elements that **survive forever**, i.e. elements that lie in the kernel of d^r for all $r \in \mathbb{N}$. The submodule $B^\infty = \bigcup_n B^n$ consists of elements that **eventually bound**. From the tower of inclusions, we have that $B^\infty \subset Z^\infty$. As it will become clear later on, the object of our ambition will be to compute the module $E^\infty := Z^\infty/B^\infty$, the page that lies at the end of an infinite sequence of successive homologies.

Definition 1.7. A spectral sequence $\{E^r_{\bullet,\bullet}, d^r\}$ is said to **collapse at the N-th term** if the differentials $d^r = 0$ for all $r \geq N$.

From the SES (I.1) the condition $d^r = 0$ implies that $Z^r = Z^{r-1}$ and that $B^r = B^{r-1}$. Thus the tower of submodules becomes:

$$B^2 \subset B^3 \subset \cdots \subset B^{N-1} \subset B^N = \cdots = B^\infty \subset Z^\infty = \cdots = Z^N \subset \cdots \subset Z^3 \subset Z^2 \subset E^2$$

and so $E^\infty = E^N$.

Example 1.8. Suppose $n_1, n_2 \in N$ and $\{E^r_{\bullet,\bullet}, d^r\}$ is a first-quadrant spectral sequence. In addition, assume that $E^2_{p,q} = \{0\}$ for $p > n_1$ or $q > n_2$. Then the spectral sequence collapses at the N-th term where $N = \min(n_1, n_2)$. Indeed there are two cases: either $n_1 \geq n_2$ or the other way round.

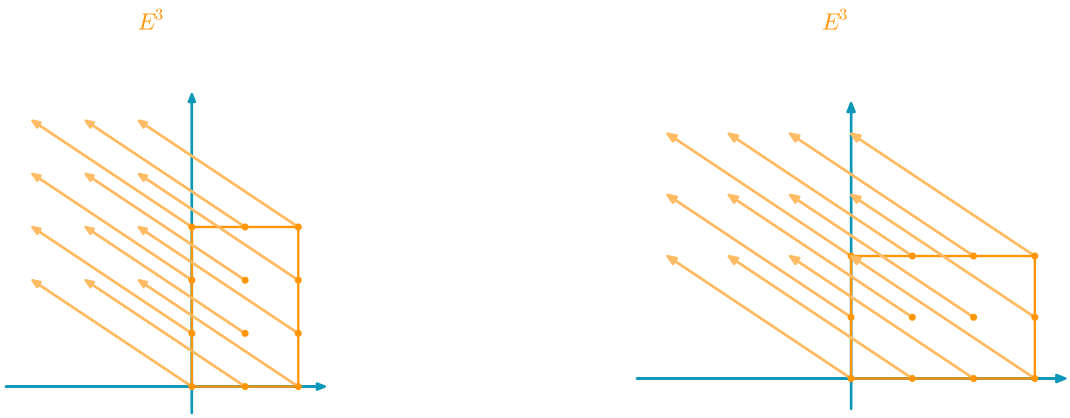


Figure I.2: Left: The case $n_1 = 2$ and $n_2 = 3$. Right: The case $n_1 = 3$ and $n_2 = 2$.

The fact that $d^r = 0$ for $r \geq N$ simply follows from the observation that the length of the arrows that represent the differentials is bigger than the boxes themselves. So we have that the target of every differential is the zero module $\{0\}$.

Having identified the limit-page, we are now able to link spectral sequences and filtered graded modules.

Definition 1.9. A spectral sequence $\{E^r, d^r\}$ is said to **converge** to $H_\bullet$, a graded R-module with filtration $F_\bullet$, if

$$E^\infty_{p,q} \cong E^0_{p,q}(H_\bullet, F_\bullet)$$

where $E^0_{p,q}(H_\bullet, F_\bullet)$ is the associated bigraded module. We will denote convergence by

$$E^1_{p,q} \Rightarrow H_{p+q}.$$

1.2 How a Spectral Sequence Arises

In the following, we will showcase one instance of how to get from algebraic data to a spectral sequence.

Definition 1.10. An R-module A is a **filtered differential graded module** if:

1. A is a direct sum of submodules, $A = \bigoplus_{n=0}^\infty A_n$.
2. There is an R-linear mapping $d : A \to A$ of degree -1, satisfying $d \circ d = 0$.
3. A has a filtration $F_\bullet$ and the differential d respects the filtration, i.e. $d : F_pA \to F_pA$.

Define $H(A, d) := \ker d / \operatorname{im} d$. It inherits a filtration through the inclusion of F_pA into A:

$$F_p(H(A, d)) = \operatorname{im}(H(F_pA, d) \xrightarrow{\text{incl.}} H(A, d)).$$

Here comes now one of the main theorems. Its proof is mostly a question of properly dealing with subscripts, and thus the theorem may be taken as a black box without losing a lot of insight into future concepts. The proof is adapted to the homological case.

Theorem 1.11 ([3, Thm. 2.6]). *Each filtered differential graded module $(A, d, F_\bullet)$ determines a spectral sequence, $\{E^r_{\bullet,\bullet}, d^r\}$, $r = 1, 2, 3, \ldots$ with d^r of bidegree $(-r, r-1)$ and*

$$E^1_{p,q} \cong H_{p+q}(F_pA / F_{p-1}A).$$

Suppose further that the filtration is bounded, that is, for each dimension n, there are values $s[n]$ and $t[n]$ such that

$$\{0\} = F_{s[n]}A_n \subset F_{s[n]+1}A_n \subset \cdots \subset F_{t[n]}A_n = A_n.$$

Then the spectral sequence converges to $H(A, d)$

$$: E^\infty_{p,q} \cong F_pH_{p+q}(A, d) / F_{p-1}H_{p+q}(A, d).$$

Proof. For the proof, keep the following in mind:

$$\dots F_{p-1}A_{p+q} \subset F_pA_{p+q} \subset F_{p+1}A_{p+q} \dots$$

as well as $d(F_pA_{p+q}) \subset F_pA_{p+q-1}$, i.e. the differential is stable. Define for all $r \geq 0$

$$\begin{aligned} Z^r_{p,q} &= F_pA_{p+q} \cap d^{-1}(F_{p-r}A_{p+q-1}) \\ B^r_{p,q} &= F_pA_{p+q} \cap d(F_{p+r}A_{p+q+1}) \\ Z^\infty_{p,q} &= \ker d \cap F_pA_{p+q} \\ B^\infty_{p,q} &= \operatorname{im} d \cap F_pA_{p+q}. \end{aligned}$$

Note that:

$$\begin{aligned} d(Z^r_{p+r,q-r+1}) &= d(F_{p+r}A_{p+q+1} \cap d^{-1}(F_pA_{p+q})) \\ &= F_pA_{p+q} \cap d(F_{p+r}A_{p+q+q}) \\ &= B^r_{p,q}. \end{aligned}$$

The assumption that the filtration is bounded implies that for $r > -s[p+q-1]+p$ and $r \geq -p + t[p+q-1]$ we have that $Z^r_{p,q} = Z^\infty_{p,q}$ and $B^r_{p,q} = B^\infty_{p,q}$. This ensures convergence and we get the following tower of inclusions

$$B^0_{p,q} \subset B^1_{p,q} \subset \cdots \subset B^{r[p,q]}_{p,q} = B^\infty_{p,q} \subset \cdots \subset B^\infty_{p,q} \subset Z^\infty_{p,q} \subset \cdots \subset Z^{r[p,q]}_{p,q} = Z^\infty_{p,q} \subset \cdots \subset Z^1_{p,q} \subset Z^0_{p,q}.$$

Define for all $0 \leq r \leq \infty$, $E^r_{p,q} := Z^r_{p,q}/(Z^{r-1}_{p-1,q+1} + B^{r-1}_{p,q})$ and the canonical projection $\eta^r_{p,q} : Z^r_{p,q} \to E^r_{p,q}$. Observe that $d(Z^r_{p,q}) = B^r_{p-r,q+r-1} \subset Z^r_{p-r,q+r-1}$ and

$$\begin{aligned} d(Z^{r-1}_{p-1,q+1} + B^{r-1}_{p,q}) &= d(Z^{r-1}_{p-1,q+1}) + d(B^{r-1}_{p,q}) \\ &\subset B^{r-1}_{p-r,q+r-1} + \{0\} \\ &\subset Z^{r-1}_{p-r-1,q+r} + B^{r-1}_{p-r,q+r-1}. \end{aligned}$$

Thus the differential as a map $d : Z^r_{p,q} \to Z^r_{p+r,q-r+1}$ induces a homomorphism d^r such that

$$\begin{array}{ccc} Z^r_{p,q} & \xrightarrow{\quad d \quad} & Z^r_{p-r,q+r-1} \\ {\scriptstyle \eta^r_{p,q}}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \eta^r_{p-r,q+r-1}} \\ E^r_{p,q} & \xrightarrow[\quad d^r \quad]{} & E^r_{p-r,q+r-1}. \end{array}$$

The operator d^r clearly inherits the property that $(d^r)^2 = 0$ from d.

With this setup out of the way, we must now establish the following assertions to complete the proof:

I. $H_\bullet(E^r_{\bullet,\bullet}, d^r) = E^{r+1}_{\bullet,\bullet}$,

II. $E^1_{p,q} \cong H_{p+q}(F_pA/F_{p-1}A)$,

III. $E^\infty_{p,q} \cong F_pH_{p+q}(A,d)/F_{p-1}H_{p+q}(A,d)$.

I. Consider the diagram

$$\begin{array}{ccccccc}
Z^r_{p-1,q+1} + B^r_{p,q} & \hookrightarrow & Z^{r+1}_{p,q} & \hookrightarrow & Z^r_{p,q} & \xrightarrow{d} & Z^r_{p-r,q+r-1} \\
 & & \downarrow{\scriptstyle \eta^r_{p,q}\big|_{Z^{r+1}_{p,q}}} & & \downarrow{\scriptstyle \eta^r_{p,q}} & & \downarrow{\scriptstyle \eta^r_{p-r,q+r-1}} \\
 & & \ker d^r & \hookrightarrow & E^r_{p,q} & \xrightarrow{d^r} & E^r_{p-r,q+r-1} \\
 & & \downarrow & & & & \\
 & & H_{p,q}(E^r_{\bullet,\bullet}, d^r) & & & & \\
 & & \downarrow & & & & \\
 & & 0. & & & &
\end{array} \tag{I.2}$$

In order to prove that this diagram commutes, we need the following lemma.

Lemma 1.12.

(a) $\eta^r_{p,q}(Z^{r+1}_{p,q}) = \ker d^r$,
(b) $Z^r_{p-1,q+1} + B^r_{p,q} = Z^{r+1}_{p,q} \cap ((\eta^r_{p,q})^{-1}(\operatorname{im} d^r))$.

Proof of Lemma 1.12.

(a) Consider $\eta^{-1}(\ker d^r)$. Since $d^r \circ \eta = \eta \circ d$, we have that for $z \in Z^r_{p,q}$,

$$\begin{aligned}
d^r(\eta^r_{p,q}(z)) = 0 \Leftrightarrow dz \in \ker \eta^r_{p-r-1,q+r} &= Z^{r-1}_{p-r-1,q+r} + B^{r-1}_{p-r,q+r-1} \\
\Leftrightarrow z \in d^{-1}(Z^{r-1}_{p-r-1,q+r} &+ B^{r-1}_{p-r,q+r-1}) \\
&= d^{-1}(F_{p-r-1}A_{p+q-1} \cap d^{-1}(d^{-1}(F_{p-2r}A_{p+q-2})) \\
&+ d^{-1}(F_{p-r}A_{p+q-1}) \cap d^{-1}(d(F_{p-1}A_{p+q})) \\
&= Z^{r+1}_{p,q} + Z^{r-1}_{p-1,q+1}.
\end{aligned}$$

Thus $\eta^{-1}(\ker d^r) = Z^{r+1}_{p,q} + Z^{r-1}_{p-1,q+1}$ and so $\ker d^r = \eta(Z^{r+1}_{p,q})$.

(b) $\operatorname{im} d^r = \eta^r_{p,q}(d(Z^r_{p+r,q-r+1})) = \eta^r_{p,q}(B^r_{p,q})$ and so

$$\begin{aligned}
(\eta^r_{p,q})^{-1}(\operatorname{im} d^r) &= B^r_{p,q} + \ker \eta^r_{p,q} \\
&= B^r_{p,q} + B^{r-1}_{p,q} + Z^{r-1}_{p-1,q+1} \\
&= B^r_{p,q} + Z^{r-1}_{p-1,q+1}.
\end{aligned}$$

Now writing out the definitions:

$$\begin{aligned}
Z^{r-1}_{p-1,q+1} \cap Z^{r+1}_{p,q} &= F_{p+1}A_{p+q} \cap F_pA_{p+q} \cap d^{-1}(F_{p-r}A_{p+q-1}) \\
&= F_{p-1}A_{p+q} \cap d^{-1}(F_{p-r-1}A_{p+q-1}) \\
&= Z^r_{p-1,q+1}.
\end{aligned}$$

Combining these two equations yields

$$Z^{r-1}_{p-1,q+1} \cap (\eta^r_{p,q})^{-1}(\operatorname{im} d^r) = Z^r_{p-1,q+1} + B^r_{p,q}.$$

□ Lemma

To conclude the proof, define $\gamma : Z^{r+1}_{p,q} \to H_{p,q}(E^r_{\bullet,\bullet}, d^r)$ to be the composition of $\eta^r_{p,q}\big|_{Z^{r+1}_{p,q}}$ and the canonical projection $\ker d^r \to H_{p,q}(E^r_{\bullet,\bullet}, d^r)$.

The kernel of γ is given by $Z^{r+1}_{p,q} \cap (\eta^r_{p,q})^{-1}(\operatorname{im} d^r)$ which is equal to $Z^r_{p-1,q+1} + B^r_{p,q}$ by part b) of the Lemma. Since γ is surjective, we have the isomorphism

$$H_{p,q}(E^r_{\bullet,\bullet}, d^r) \cong Z^{r+1}_{p,q}/(Z^r_{p-1,q+1} + B^r_{p,q}).$$

II. Define

$$Z^{-1}_{p-1,q+1} := F_{p-1}A_{p+q} \text{ and } B^{-1}_{p-1,q+1} := d(F_{p-1}A_{p+q+1}).$$

Observe that $E^0_{p,q} = Z^0_{p,q}/(Z^{-1}_{p-1,q+1} + B^{-1}_{p,q})$. Since d respects the filtration

$$\begin{aligned} E^0_{p,q} &= F_pA_{p+q} \cap d^{-1}(F_pA_{p+q-1}/F_{p-1}A_{p+q} + d(F_{p-1}A_{p+q+1}) \\ &= F_pA_{p+q}/F_{p-1}A_{p+q}. \end{aligned}$$

The differential $d^0 : E^0_{p,q} \to E^0_{p,q-1}$ is induced by the differential $d : F_pA_{p+q} \to F_pA_{p+q-1}$. This implies that

$$E^1_{p,q} \cong H_{p+q}(F_pA/F_{p-1}A).$$

III. Let $\eta^\infty_{p,q} : Z^\infty_{p,q} \to E^\infty_{p,q}$ and let $\pi : \ker d \to H(A,d)$ denote the canonical projection. Then:

$$F_pH_{p+q}(A,d) = H_{p+q}(\operatorname{im}(F_pA \to A), d) = \pi(F_pA_{p+q} \cap \ker d) = \pi(Z^\infty_{p,q}).$$

Since $\pi(\ker \eta^\infty_{p,q}) = \pi(Z^\infty_{p-1,q+1} + B^\infty_{p,q}) = F_{p-1}H_{p+q}(A,d)$ we have that π induces a map $d^\infty : E^\infty_{p,q} \to F_pH_{p+q}(A,d)/F_{p+1}H_{p+q}(A,d)$. With kernel

$$\begin{aligned} \ker d^\infty &= \eta^\infty_{p,q}(\pi^{-1}(F_{p-1}H_{p+q}(A,d) \cap Z^\infty_{p,q})) \\ &= \eta^\infty_{p,q}(Z^\infty_{p-1,q+1} \cap d(A) \cap Z^\infty_{p,q}) \\ &\subset \eta^\infty_{p,q}(Z^\infty_{p-1,q+1} + B^\infty_{p,q}) = \{0\}. \end{aligned}$$

Thus we conclude that d^∞ is an isomorphism.

□

1.3 Spectral Sequences of a Double Complex

Before moving on to the more topological part, we need one more algebraic result. In the following we introduce the notion of a double complex.

In section 2, we will construct a double complex from which the Serre spectral sequence derives.

Definition 1.13. A **double complex** $\{M_{\bullet,\bullet}, d', d''\}$ is a bigraded module $M_{\bullet,\bullet}$, together with two bigraded maps $d' : M_{\bullet,\bullet} \to M_{\bullet,\bullet}$ and $d'' : M_{\bullet,\bullet} \to M_{\bullet,\bullet}$, where d' is of bidegree $(-1, 0)$ and d'' is of bidegree $(0, -1)$.In addition, the maps should satisfy $d' \circ d' = 0$, respectively $d'' \circ d'' = 0$ and $d' \circ d'' + d'' \circ d' = 0$.

We associate to each double complex its **total complex**, $\mathrm{tot}(M)_\bullet$, which is the differential graded module over R defined by

$$\mathrm{tot}(M)_n = \bigoplus_{p+q=n} M_{p,q}$$

with boundary operator $d := d' + d''$.

Definition 1.14. Let $\{M_{\bullet,\bullet}, d', d''\}$ be a double complex. The **transpose** of the double complex is $\{{}^\top M_{\bullet,\bullet}, {}^\top d', {}^\top d''\}$, where ${}^\top M_{p,q} = M_{q,p}$, ${}^\top d'_{p,q} = d''_{q,p}$ and ${}^\top d''_{p,q} = d'_{q,p}$. Then $\{{}^\top M_{\bullet,\bullet}, {}^\top d', {}^\top d''\}$ is a double complex such that $\mathrm{tot}({}^\top M) = \mathrm{tot}(M)$.

The crux of double complexes is that one can take the homology of $M_{\bullet,\bullet}$ in two different directions. We will use this fact to construct two spectral sequences that both converge to $H(\mathrm{tot}(M), d)$.

Definition 1.15. Let $\{M_{\bullet,\bullet}, d', d''\}$ be a double complex. Then for fixed p, respectively q, we have that $(M_{\bullet,q}, d')$ and $(M_{p,\bullet}, d'')$ both form complexes. Define the homology of the rows to be the bigraded module $H'_{\bullet,\bullet}(M)$ such that:

$$H'_{p,q}(M) = \ker d'_{p,q} / \operatorname{im} d'_{p+1,q}.$$

Analogously, define the homology of the columns $H''_{\bullet,\bullet}(M)$:

$$H''_{p,q}(M) = \ker d''_{p,q} / \operatorname{im} d''_{p,q+1}.$$

Construction 1.16. Fix $q \in \mathbb{Z}$, then the q-th row of $H''_{\bullet,\bullet}(M)$ can be made into a chain complex of its own by combining it with the induced boundary map:

$$\begin{aligned} \bar{d}' : H''_{p,q} &\to H''_{p-1,q}(M) \\ z + \operatorname{im} d''_{p,q+1} &\mapsto d'_{p,q}(z) + \operatorname{im} d'_{p-1,q+1}. \end{aligned}$$

Using the definitions, one can easily check that this map is well-defined and that $\bar{d}' \circ \bar{d}' = 0$. With this data, define the **first iterated homology** of the double complex, denoted $H'_\bullet H''_\bullet(M)$, by setting:

$$H'_p H''_q(M) := H_p(H''_{\bullet,q}(M), \bar{d}') = \ker \bar{d}'_{p,q} / \operatorname{im} \bar{d}'_{p+1,q}.$$

Similarly, define the **second iterated homology** $H''_\bullet H'_\bullet(M)$ as:

$$H''_p H'_q(M) := H_p(H'_{q,\bullet}(M), \bar{d}'') = \ker \bar{d}''_{p,q} / \operatorname{im} \bar{d}''_{p,q+1}$$

where $\bar{d}''$ is defined in the same way.

We now move on to extracting spectral sequences from double complexes.

DEFINITION 1.17. Let $\{M_{\bullet,\bullet}, d', d''\}$ be a double complex. Define the **column-wise filtration** of $\mathrm{tot}(M)$ as:

$$^{I}(F_p \,\mathrm{tot}(M))_n := \bigoplus_{i \leq p} M_{i,n-i}.$$

Symmetrically, define the **row-wise filtration** as:

$$^{II}(F_p \,\mathrm{tot}(M))_n := \bigoplus_{j \leq p} M_{n-j,j}.$$

Plugging these two filtrations into Theorem 1.11 produces the following result:

THEOREM 1.18 ([3, Thm. 2.15]). *Given a double complex $\{M_{\bullet,\bullet}, d', d''\}$, there are two spectral sequences $\{^{I}E^r, {}^{I}d^r\}$ and $\{^{II}E^r, {}^{II}d^r\}$ with first pages:*

$$^{I}E^1_{p,q} \cong H''_{p,q}(M), \text{ and } {}^{II}E^1_{p,q} \cong H'_{p,q}(M)$$

and second pages:

$$^{I}E^2_{p,q} \cong H'_p H''_q(M), \text{ and } {}^{II}E^2_{p,q} \cong H''_p H'_q(M).$$

Moreover, if $M_{p,q} = \{0\}$ when $p < 0$ or $q < 0$, then both sequences are first quadrant, and converge to the homology of the total complex $H_\bullet(\mathrm{tot}(M)_\bullet, d)$:

$$^{I}E^1_{p,q} \Rightarrow H_{p+q}(\mathrm{tot}(M)_\bullet) \text{ , and } {}^{II}E^1_{p,q} \Rightarrow H_{p+q}(\mathrm{tot}(M)_\bullet).$$

Proof. We are only going to prove the existence of $\{^{I}E^r, {}^{I}d^r\}$, the other spectral sequence follows by symmetry.

Since $M_{p,q} = \{0\}$ when $p < 0$ or $q < 0$, the filtration $^{I}(F_p \,\mathrm{tot}(M))_n = \bigoplus_{i \leq p} M_{i,n-1}$ is bounded from above and below. Thus we are able to apply Theorem 1.11 to obtain a spectral sequence $\{^{I}E^r, {}^{I}d^r\}$, with:

$$^{I}E^1_{p,q} = H_{p+q}((^{I}(F_p \,\mathrm{tot}(M))/{}^{I}(F_{p-1} \,\mathrm{tot}(M)))_\bullet, d)$$

where $d = d' + d''$ is the boundary operator of the total homology. The first differential is given by:

$$^{I}d^1 : H_{p+q}\left(\left(\frac{^{I}F_p \,\mathrm{tot}\, M}{^{I}F_{p-1} \,\mathrm{tot}\, M}\right)_\bullet\right) \to H_{p+q-1}\left(\left(\frac{^{I}F_{p-1} \,\mathrm{tot}\, M}{^{I}F_{p-2} \,\mathrm{tot}\, M}\right)_\bullet\right)$$
$$[z + (^{I}F_{p-1} \,\mathrm{tot}\, M)_{p+q}] \mapsto [dz + (^{I}F_{p-2} \,\mathrm{tot}\, M)_{p+q-1}]$$

Furthermore, boundedness of the filtration implies that $\{^{I}E, {}^{I}d^r\}$ converges to $H_\bullet(\mathrm{tot}\, M)$, or explicitly:

$$^{I}E^\infty_{p,q} \cong \frac{^{I}F_p(H_{p+q}(\mathrm{tot}\, M))}{^{I}F_{p-1}(H_{p+q}(\mathrm{tot}\, M))}.$$

Consider the $(p+q)$-th term of the quotient complex given by:

$$\frac{^{I}F_p(H_{p+q}(\mathrm{tot}\, M))}{^{I}F_{p-1}(H_{p+q}(\mathrm{tot}\, M))} = \frac{\bigoplus_{i \leq p} M_{i,p+q-i}}{\bigoplus_{i \leq p-1} M_{i,p+q-i}} \cong M_{p,q}.$$

And thus each element of the quotient group $a \in \frac{{}^I F_p(H_{p+q}(\operatorname{tot} M))}{{}^I F_{p-1}(H_{p+q}(\operatorname{tot} M))}$ may be represented by a unique element $a_{p,q} \in M_{p,q}$.

By a slight abuse of notation, let $d : \frac{{}^I F_p(\operatorname{tot} M)_{p+q}}{{}^I F_{p-1}(\operatorname{tot} M)_{p+q}} \to \frac{{}^I F_p(\operatorname{tot} M)_{p+q-1}}{{}^I F_{p-1}(\operatorname{tot} M)_{p+q-1}}$ denote the induced boundary operator on the quotient complex. Then for $a = a_{p,q} + \bigoplus_{i \leq p-1} M_{i,p+q-i}$, we have that $d(a) = d(a_{p,q}) + ({}^I F_{p-1} \operatorname{tot} M)_{p+q-1}$.

Now $d(a_{p,q}) = d'(a_{p,q}) + d''(a_{p,q}) \in M_{p-1,q} \oplus M_{p,q-1}$ but $M_{p-1,q} \subset^I F_{p-1}(\operatorname{tot} M)_{p+q-1}$ and thus

$$d(a) = d''(a_{p,q}) +^I F_{p-1}(\operatorname{tot} M)_{p+q-1}.$$

This shows that the following diagram commutes

$$\begin{array}{ccc} \frac{{}^I F_p(\operatorname{tot} M)_{p+q}}{{}^I F_{p-1}(\operatorname{tot} M)_{p+q}} & \xrightarrow{\cong} & M_{p,q} \\ \Big\downarrow{\scriptstyle d} & & \Big\downarrow{\scriptstyle \partial''} \\ \frac{{}^I F_p(\operatorname{tot} M)_{p+q-1}}{{}^I F_{p-1}(\operatorname{tot} M)_{p+q-1}} & \xrightarrow[\cong]{} & M_{p,q-1} \end{array}$$

implying that $H''_{p,q}(M) \cong H_{p+q}\left(\left(\frac{{}^I F_p \operatorname{tot} M}{{}^I F_{p-1} \operatorname{tot} M}\right)_\bullet\right)$. Call this isomorphism Φ.

To prove that ${}^I E^2_{p,q} \cong H'_p H''_q(M)$, recall that $H'_p H''_q(M) = \ker \bar{d}'_{p,q} / \ker \bar{d}'_{p+1,q}$. Let $[z] \in H''_{p,q}(M)$ with $z \in \ker d''_{p,q}$. Then

$$\Phi([z]) = [z + ({}^I F_{p-1} \operatorname{tot} M)_{p+q}].$$

By the definition of ${}^I d^1$, we have

$${}^I d^1(\Phi([z])) = [dz + ({}^I F_{p-2} \operatorname{tot} M)_{p+q-1}] = [d'z + ({}^I F_{p-2} \operatorname{tot} M)_{p+q-1}]$$

since $d''z = 0$. Applying Φ^{-1} to this element yields $[d'z] \in H''_{p-1,q}(M)$ which is equal to $\bar{d}'([z])$, thus proving that the following commutes

$$\begin{array}{ccc} H''_{p,q}(M) & \xrightarrow{\Phi} & H_{p+q}\left(\left(\frac{{}^I F_p(\operatorname{tot} M)}{{}^I F_{p-1}(\operatorname{tot} M)}\right)_\bullet\right) \\ \Big\downarrow{\scriptstyle \bar{d}'} & & \Big\downarrow{\scriptstyle {}^I d^{1\prime\prime}} \\ H''_{p-1,q}(M) & \xrightarrow[\Phi]{} & H_{p+q}\left(\left(\frac{{}^I F_p(\operatorname{tot} M)}{{}^I F_{p-1}(\operatorname{tot} M)}\right)_\bullet\right). \end{array}$$

Therefore ${}^I E^2_{p,q} \cong H'_p H''_q(M)$. □

Remark 1.19. Although we are only going to directly use double complexes to derive the Leray-Serre spectral sequence, we state the following important application. Recall that the Künneth formula for homology states that given X and Y topological spaces, and R a principal ideal domain, there are natural short sequences

$$0 \to \bigoplus_{i+j=k} H_i(X;R) \otimes_R H_j(Y;R) \to H_k(X \times Y; R) \to \bigoplus_{i+j=k-1} \operatorname{Tor}_1^R(H_i(X;R), H_j(Y;R)) \to 0.$$

For a general commutative ring R, one can use double complexes to construct a spectral sequence that converges to $H_{p+q}(X \times Y; R)$, namely:

$$E^2_{p,q} = \bigoplus_{q_1+q_2=q} \operatorname{Tor}_p^R(H_{q_1}(X;R), H_{q_2}(Y;R)) \Rightarrow H_{p+q}(X \times Y; R).$$

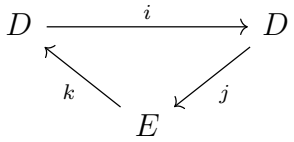
See for example [3, p.50] for more details.

1.4 Exact Couples

There is another setting in which spectral sequences arise, the so-called exact couples. One advantage over the approach of constructing a spectral sequence with the aid of filtrations is that exact couples are comparatively easy to describe.

On the other hand, one downside is that in the case of convergence, the target is sometimes harder to identify. We are mainly going to focus on spectral sequences of cohomological type as we are only going to use exact couples in the second chapter that deals with cohomology.

Definition 1.20. Let D and E denote R-modules and let $i : D \to D$, $j : D \to E$ and $k : E \to D$ be module homomorphisms. We present the data in the following (not necessarily commutative) diagram:



Call $\mathcal{C} := \{D, E, i, j, k\}$ an **exact couple** if this diagram is exact at each module, i.e. $\operatorname{im} i = \ker j$, $\operatorname{im} j = \ker k$ and $\operatorname{im} k = \ker i$.

By setting $d = j \circ k : E \to E$, one can see that E is a differential R-module, since $d \circ d = 0$. From this exact couple, one derives a new one:

$$E' := H(E, d) = \ker d / \operatorname{im} d, \qquad D' := i(D) = \ker j$$

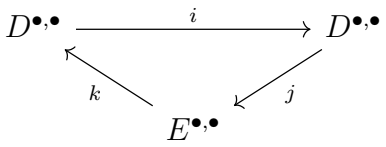
Define the maps $i' = i_{|i(D)} : D' \to D'$, $j' : D' \to E'$, $j'(i(x)) = j(x) + dE$ and $k' : E' \to D'$, $k'(e + dE) = k(e)$. It is easy to check that these maps are well-defined.

We call $\mathcal{C}' := \{D', E', i', j', k'\}$ the derived couple. An application of the definitions then yields that $\mathcal{C}'$ is an exact couple. Iterating this process, we obtain the n-th derived couple of $\mathcal{C}$:

$$\mathcal{C}^{(n)} := \{D^{(n)}, E^{(n)}, i^{(n)}, j^{(n)}, k^{(n)}\} = (\mathcal{C}^{(n-1)})'.$$

By introducing a bigrading of the involved modules, we are able to link exact couples to spectral sequences:

Theorem 1.21 ([3, Thm. 2.8]). *Suppose $D^{\bullet,\bullet} = \{D^{p,q}\}$ and $E^{\bullet,\bullet} = \{E^{p,q}\}$ are bigraded modules over R together with homomorphisms i of bidegree $(-1, 1)$, j of bidegree $(0, 0)$ and k of bidegree $(1, 0)$*



This data determines a spectral sequence $\{E_r, d_r\}$ *for* $r = 1, 2, \ldots$ *of cohomological type, with* $E_r = (E^{\bullet,\bullet})^{(r-1)}$*, the* $(r-1)$*-st derived module of* $E^{\bullet,\bullet}$ *and* $d_r = j^{(r)} \circ k^{(r)}$*.*

To show the link between exact couples and filtrations, we are going to state the equivalence of both approaches. Let (A, d, F) be a (cohomologically) filtered differential graded R-module. For each filtration degree p, there is a short exact sequence:

$$0 \longrightarrow F_{p+1}A \longrightarrow F_pA \longrightarrow F_pA/F_{p+1}A \longrightarrow 0.$$

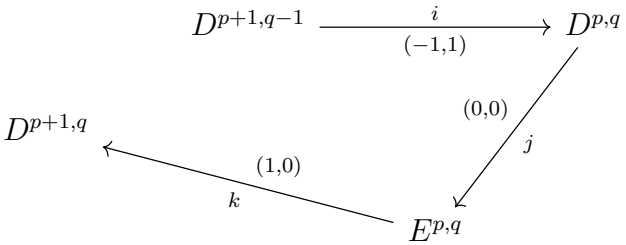
Applying the cohomology functor, we obtain a long exact sequence:

$$\cdots \xrightarrow{k} H^{p+q}(F^{p+1}A) \xrightarrow{i} H^{p+q}(F^pA) \xrightarrow{j} H^{p+q}(F^pA/F^{p+1}A) \xrightarrow{k} \cdots$$

where k is the connecting homomorphism. Define the bigraded modules

$$E^{p,q} = H^{p+q}(F^pA/F^{p+1}A) \text{ and } D^{p,q} = H^{p+q}(F^pA).$$

This yields an exact couple



where we added the bidegrees. Using Theorem 1.21 we obtain a spectral sequence and have the following equivalence result:

Proposition 1.22 ([3, Prop. 2.11]). *For a filtered differential graded* R*-module* (A, d, F)*, the spectral sequence associated to the decreasing filtration and the spectral sequence associated to the exact couple are the same.*

2 The Serre Spectral Sequence

In this section we give the precise statement of the Serre spectral sequence. We first define what a Serre fibration is, and then prove several important results about them. We then move on to define homology with local coefficients, a tool that allows us to encode the "twistedness" of non-trivial fibrations. Last but not least, we present a construction of the Serre spectral sequence that is due to Andreas Dress.

2.1 Some Topology

Definition 2.1. A mapping $p : E \to B$ has the **homotopy lifting property** (HLP) with respect to a space X if, given a homotopy $H : X \times I \to B$ and a mapping $h : X \times \{0\} \to E$ such that $p \circ h(y, 0) = H(y, 0)$, there exists a homotopy $\widetilde{H} : X \times I \to E$ such that $\widetilde{H}(y, 0) = h(y, 0)$ and $p \circ \widetilde{H} = H$. Pictorially the following commutes:

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{h} & E \\ \downarrow & \overset{\widetilde{H}}{\nearrow} & \downarrow p \\ X \times I & \xrightarrow[H]{} & B. \end{array}$$

Define $F := p^{-1}(x)$ for $x \in B$. The space F will be called a **fibre** of p. A continuous map with the HLP with respect to all spaces is called a **fibration**. A continuous map that satisfies the HLP with respect to cubes I^n is called a **Serre fibration**. Sometimes we will denote fibrations by

$$F \to E \to B.$$

It can be shown that if B is path-connected all the fibres are homotopy equivalent.

Example 2.2.

1. Let X and F be topological spaces, define $E := X \times F$ and let $p : E \to X$ denote the projection onto the first factor. Then p is a fibration with fibre F.

2. The Hopf fibration $S^1 \to S^3 \to S^2$ was one of the historically first non-trivial examples of a fibration.

3. Fibre bundles over paracompact base spaces are fibrations. This yields an abundance of additional examples. The **Möbius strip** is a fibration over the circle S^1 with fibre $\mathbb{R}$. Another fibre over the circle is the **Klein bottle**, but this time with fibre S^1.

A further fruitful source of fibrations comes from Differential Geometry. Here are a few major results.

Theorem 2.3 ([6, Corr. 24.9]). *Let G be a Lie group and $H \subset G$ a closed subgroup. Let G/H denote the set of left cosets of H. Then $\pi : G \to G/H$ is a fibration (more specifically a principal H-bundle).*

We can also have Lie groups acting on manifolds.

DEFINITION 2.4. Let $\mu : G \times M \to M$ be a left action of a Lie group G on a smooth manifold M. The action μ is said to be transitive if for any two points $x, y \in M$, there exists $a \in G$ such that $\mu(a, x) = y$.

Suppose now that $\mu : G \times M \to M$ is a left action. Fix a point $x \in M$, and set $H := \{a \in G : \mu(a, x) = x\}$. Then H is a closed subgroup of G, and the action of G restricted to H gives an action of H on M for which x is a fixed point. We call H the **isotropy group** of x.

THEOREM 2.5 ([6, Thm. 12.11]). *Let $\mu : G \times M \to M$ be a transitive left action of a Lie group G on a smooth manifold M. Fix $x \in M$ and let H denote the isotropy group of x. Let $\pi : G \to G/H$ denote the quotient map. Define*

$$\begin{aligned} \varphi : G/H &\to M \\ \pi(a) &\mapsto \mu(a, x). \end{aligned}$$

Then φ is a diffeomorphism.

LEMMA 2.6. *There is a fibration*

$$U(n-1) \to U(n) \to S^{2n-1}$$

where $U(n) := GL(n; \mathbb{C})$ denotes the unitary group.

Proof. Define

$$\begin{aligned} \mu : U(n) \times S^{2n-1} &\to S^{2n-1} \\ (A, x) &\mapsto Ax \end{aligned}$$

This action is transitive, since for every point $x \in S^{2n-1} \subset \mathbb{C}^n$ there exists a rotation $A \in U(n)$ such that $Ax = (0, \dots, 0, 1) =: e_n$.

Moreover the isotropy group of e_n is given by those matrices $A \in U(n)$ of the form

$$A = \begin{pmatrix} & & 0 \\ & B & \vdots \\ & & 0 \\ 0 & \dots & 0 \quad 1 \end{pmatrix}$$

where $B \in U(n-1)$. We conclude by Theorem 2.5 that $S^{2n-1} \cong U(n)/U(n-1)$. To finish the proof, we use Theorem 2.3 to deduce that

$$U(n-1) \to U(n) \to U(n)/U(n-1) \cong S^{2n-1}$$

is a fibration. □

We interrupt our exposition for an interlude on point-set topology. Sometimes it will be necessary to topologise the space of continuous maps between two topological spaces.

DEFINITION 2.7. Let X and Y denote topological spaces, and write $\mathrm{Hom}(X,Y)$ for the set of all continuous functions from X to Y. Define a topology on $\mathrm{Hom}(X,Y)$, called the **compact-open topology** by declaring that a sub-basis is all sets of the form

$$(K;U) := \{f \in \mathrm{Hom}(X,Y) \mid f(K) \subset U\},$$

where $K \subset X$ is compact and U is open.

DEFINITION 2.8. Let X and Y be topological spaces. Define the **evaluation map**

$$\mathrm{ev} : \mathrm{Hom}(X,Y) \times X \to Y$$

by

$$\mathrm{ev}(f,x) := f(x).$$

One nice property of the compact-open topology is the following:

PROPOSITION 2.9. *Let X and Y be topological spaces, and assume that X is locally compact and Hausdorff. Then* $\mathrm{ev} : \mathrm{Hom}(X,Y) \times X \to Y$ *is a continuous map.*

Next up, we introduce some notation that will be useful later on.

DEFINITION 2.10. Suppose X, Y and Z are sets. If $\zeta : Z \times Y \to X$ is an element of $\mathrm{Hom}(Z \times Y, X)$, define

$$\zeta^{\#} : Z \to \mathrm{Hom}(Y,X), \qquad \zeta^{\#}(z)(y) := \zeta(z,y).$$

Similarly, if $\varphi : Z \to \mathrm{Hom}(Y,X)$, define

$$\varphi^{\flat} : Z \times Y \to X, \qquad \varphi^{\flat}(z,y) := \varphi(z)(y).$$

The next result holds trivially in the category **Sets**, but proving that the result also holds for topological spaces requires some work and additional assumptions.

THEOREM 2.11 (Exponential Law in topology). *Let X, Y and Z be topological spaces. Assume that Y is locally compact and Hausdorff. Then the operation $\zeta \mapsto \zeta^{\#}$ defines a bijection*

$$\mathrm{Hom}(Z \times Y, X) \to \mathrm{Hom}(Z, \mathrm{Hom}(Y,X)),$$

with inverse given by $\varphi \mapsto \varphi^{\flat}$. Furthermore, if in addition Z is locally compact and Hausdorff, the above is also a homeomorphism.

COROLLARY 2.12. *Let X, Y and Z be topological spaces, and assume that Y is locally compact and Hausdorff. A function $\varphi : Z \to \mathrm{Hom}(Y,X)$ is continuous if and only if the composition $\mathrm{ev} \circ (\varphi \times \mathrm{id}_Y)$ is continuous:*

$$Z \times Y \xrightarrow{\varphi \times \mathrm{id}_Y} \mathrm{Hom}(X,Y) \times Y \xrightarrow{\mathrm{ev}} X.$$

REMARK 2.13. Cubes I^n are locally compact and Hausdorff. They thus lend themselves to an application of the exponential law.

The following is a standard application of the universal property of the pullback.

PROPOSITION 2.14. *The pullback of a Serre fibration is a Serre fibration.*

We now prove some properties of maps of cubes into the total or base space.

PROPOSITION 2.15. *If $p : E \to B$ is a Serre fibration, then the induced continuous map*

$$\begin{aligned} \hat{p} : \mathrm{Hom}(I^s, E) &\to \mathrm{Hom}(I^s, B) \\ f &\mapsto p \circ f \end{aligned}$$

is also a Serre fibration.

Proof. To prove this we are going to use the musical bijections defined in Definition 2.10. Let $H : I^n \times I \to \mathrm{Hom}(I^s, B)$ be a homotopy and $h : I^n \times \{0\} \to \mathrm{Hom}(I^s, E)$ be a continuous map such that $p \circ h(a, 0) = H(a, 0)$. The map $H^\flat : I^n \times I^s \times I \to B$ is a homotopy from the cube I^{n+s} to the base space B and $h^\flat : I^{n+s} \times \{0\} \to E$ a continuous map compatible with the projection p. Thus we can apply the HLP to obtain a map $\widetilde{H^\flat} : I^n \times I^s \times I \to E$ that makes the following diagram commute:

$$\begin{array}{ccc} I^n \times I^s \times \{0\} & \xrightarrow{h^\flat} & E \\ \downarrow & \overset{\widetilde{H^\flat}}{\nearrow} & \downarrow p \\ I^n \times I^s \times I & \xrightarrow[H^\flat]{} & B. \end{array}$$

The map $(\widetilde{H^\flat})^\#$ then solves the lifting problem. Indeed, given $a \in I^n, t \in I$ and $b \in I^s$, we obtain

$$(\hat{p} \circ (\widetilde{H^\flat})^\#)(a, t)(b) = p(\widetilde{H^\flat}(a, t, b)) = H^\flat(a, t, b) = H(a, t)(b)$$

and

$$h(a, 0)(b) = h^\flat(a, 0, b) = \widetilde{H^\flat}(a, 0, b) = (\widetilde{H^\flat})^\#(a, 0)(b).$$

□

PROPOSITION 2.16. *For any topological space X, the continuous map $i : X \to \mathrm{Hom}(I^n, X)$ which maps each element x to the constant map c_x is a homotopy equivalence, with homotopy inverse $j : \mathrm{Hom}(I^n, X) \to X$ which maps σ to $\sigma(0)$.*

Proof. Obviously, $ji = \mathrm{id}_X$. To prove that $ij \cong \mathrm{id}_{\mathrm{Hom}(I^n, X)}$ we need to show that there exists a homotopy

$$\begin{aligned} H : \mathrm{Hom}(I^n, X) \times I &\to \mathrm{Hom}(I^n, X) \\ (\sigma, 0) &\mapsto c_{\sigma(0)} \\ (\sigma, 1) &\mapsto \sigma. \end{aligned}$$

By the exponential law, this is equivalent to constructing a continuous map $H^\flat : \mathrm{Hom}(I^n, X) \times I^n \times I \to X$ with

$$H^\flat(\sigma, a, 0) = \sigma(0) \text{ and } H^\flat(\sigma, a, 0) = \sigma(a). \tag{I.3}$$

Since I^n is contractible, there is a homotopy $G : I^n \times I \to I^n$ with $G(a, 0) = 0$ and $G(a, 1) = a$. Define $\hat{H}(\sigma, s, t) := \sigma(G(s, t))$. The map $\hat{H}$ is continuous by Corollary 2.12 and satisfies equation (I.3). Consequently $(\hat{H})^\#$ is our desired homotopy. □

We will also need the following standard result from homological algebra:

PROPOSITION 2.17 (The 5-Lemma, [11, Prop. 2.1.6]). *Consider the category* ***Ab*** *of abelian groups. Suppose the following commutative diagram :*

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{j} & E & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \\ \ldots & \longrightarrow & A' & \xrightarrow{r} & B' & \xrightarrow{s} & C' & \xrightarrow{t} & D' & \xrightarrow{u} & E & \longrightarrow & \ldots \end{array}$$

where rows are exact sequences, m and p isomorphims, l surjective, and q injective. Then n is an isomorphism.

REMARK 2.18. The proposition still holds true for non abelian groups. Moreover if one goes over the proof, one notices that the only group structures that were used are the ones of the groups C, C' and D, with homomorphisms n and t. All the other groups and homomorphisms can be replaced by pointed sets and pointed set maps respectively, where l and q are no longer isomorphisms, but bijections. The "kernel" of a pointed set map is then defined as the (pointed) set of all points of the domain which are mapped into the basepoint of the target.

Serre fibrations have the following nice property that they induce a long exact sequence of the homotopy groups:

THEOREM 2.19 ([11, Thm. 2.1.2]). *Let $p : E \to B$ be a Serre fibration, equipped with basepoints $e_0 \in E$, $b_0 \in B$ such that $p(e_0) = b_0$, so that the fibre is defined by $F := p^1(b_0)$. Then there is an exact sequence :*

$$\ldots \longrightarrow \pi_{n+1}(B) \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \longrightarrow \ldots$$

PROPOSITION 2.20. *Let $p : E \to B$ be a Serre fibration, and suppose that $\gamma : I \to B$ is a path from b_0 to b_1 in B. Then $p^{-1}(b_0)$ and $p^{-1}(b_1)$ are weakly equivalent.*

Proof. Since the only fibres that we are interested in are the ones over b_0 and b_1, consider the pullback:

$$\begin{array}{ccc} E_\gamma & \longrightarrow & E \\ {\scriptstyle p_\gamma}\downarrow & & \downarrow{\scriptstyle p} \\ I & \xrightarrow[\gamma]{} & B. \end{array}$$

Then by Proposition 2.14, the map $p_\gamma : E_\gamma \to I$ is a Serre fibration, with fibres $(p_\gamma)^{-1}(\gamma(0)) = p^{-1}(b_0)$ and $(p_\gamma)^{-1}(\gamma(1)) = p^{-1}(b_1)$. But now since I is contractible, the long exact homotopy sequence of Theorem 2.19 applied to p_γ yields that

$$\pi_n(p^{-1}(b_0)) \cong \pi_n(E_\gamma) \cong \pi_n(p^{-1}(b_1)) \qquad \forall n \geq 0.$$

This concludes the proof. □

To end this subsection, we are going to give a very general result for pullbacks:

PROPOSITION 2.21. *Let $\mathcal{C}$ be a fixed category. Consider a pullback square:*

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle b} \\ D & \xrightarrow{c} & C. \end{array}$$

Given another commutative diagram (a priori not necessarily a pullback square):

$$\begin{array}{ccc} E & \xrightarrow{e} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle d} \\ F & \xrightarrow{g} & D \end{array}$$

we have that the latter diagram is a pullback if and only if the following diagram is a pullback:

$$\begin{array}{ccc} E & \xrightarrow{ae} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle b} \\ F & \xrightarrow{cg} & C. \end{array}$$

The proof follows again immediately from the definition of a pullback.

2.2 Local Systems

Fibrations are one example where we patch together local topological data, the fibres, to obtain a global structure, the total space. Relating the (co)homology of the total and base space must somehow involve lifting maps, but these lifts are then highly dependent on the local expressions of the projections onto the base. In contrary to singular (co)homology, we need a coefficient system that is more sensitive to which fibre we are currently traversing.

Definition 2.22. A **local system** $\underline{A} = \{A_x, \tau_\gamma\}$ of abelian groups on a topological space X consists of two maps: the first assigns to each point x of X an abelian group A_x, and the second assigns to each path $\gamma : \Delta^1 \to X$ (regarded as a singular 1-simplex) of X a group homomorphism $\tau_\gamma : A_{\gamma(e_1)} \to A_{\gamma(e_0)}$ such that:

1. if γ is constant, then τ_γ is the identity map,
2. for each singular 2-simplex $h : \Delta^2 \to X$, there is an identity:

$$\tau_{h\varepsilon_0}\tau_{h\varepsilon_2} = \tau_{h\varepsilon_1}.$$

If $A_x = A$ for all $x \in X$, where A is a fixed abelian group, and if all of the maps τ_γ are the identity map, then we say that $\underline{A}$ is a **constant local system with value** A.

The following lemma shows that local systems behave nicely with respect to homotopies.

Lemma 2.23 ([11, Lem. 2.2.2]). *Let $\underline{A} = \{A_x, \tau_\gamma\}$ be a local system on X. If two paths λ and λ' in X are homotopic relative to the boundary: $\lambda \simeq_* \lambda'$, then $\tau_\lambda = \tau_{\lambda'}$.*

LEMMA 2.24. *Let $\underline{A} = \{A_x, \tau_\gamma\}$ be a local system on X. Let γ_1 and γ_2 be two paths on X, such that $\gamma_1(1) = \gamma_2(0)$. Then $\tau_{\gamma_1 * \gamma_2} = \tau_{\gamma_2} \circ \tau_{\gamma_1}$.*

The construction used in the following is taken from [7, Prop. 9.3].

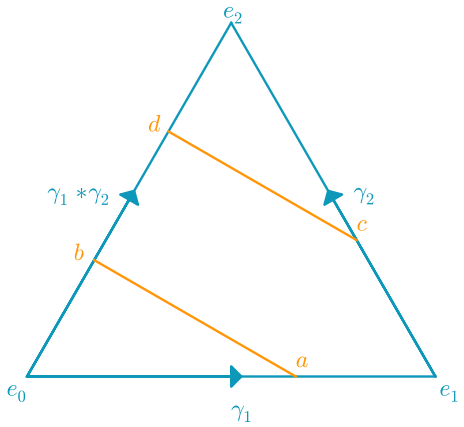
Proof. We are going to construct a continuous map $\sigma : \Delta^2 \to X$ such that:

$$\sigma \circ \varepsilon_0 = \gamma_1 \ , \ \sigma \circ \varepsilon_2 = \gamma_2 \text{ and } \sigma \circ \varepsilon_1 = -\gamma_1 * \gamma_2.$$

On the boundary, define:

$$\sigma(1-s, s, 0) = \gamma_1(s), \qquad \sigma(0, 1-s, s) = \gamma_2(s), \qquad \sigma(1-s, 0, s) = \gamma_1 * \gamma_2(s).$$

Using the figure below as visual aid, for fixed $s \in [0,1]$ we define σ on the interior of Δ^2 to be constant along the orange lines from $a = a(s)$ to $b = b(s)$, respectively constant from $c = c(s)$ to $d = d(s)$.



Explicitly, we have that $a(s) = (1-s, s, 0)$, $b(s) = (\frac{1}{2}(2-s), 0, \frac{s}{2})$, $c(s) = (0, 1-s, s)$ and $d(s) = (\frac{1}{2}(1-s), 0, \frac{1}{2}(1+s))$. To check that this is well-defined, notice that $\sigma(d(s)) = \sigma(\frac{1}{2}(1-s), 0, \frac{1}{2}(1+s)) = (\gamma_1 * \gamma_2)(\frac{1}{2} + \frac{s}{2}) = \gamma_2(s) = \sigma(c(s))$.

The Gluing Lemma then shows that σ is continuous, and hence is a singular 2-simplex on X. To conclude, observe that:

$$\partial\sigma = \sigma \circ \varepsilon_0 - \sigma \circ \varepsilon_1 + \sigma \circ \varepsilon_2 = \gamma_2 - \gamma_1 * \gamma_2 + \gamma_1.$$

□

The following proposition follows readily from the above lemmas and the definition of a local system:

PROPOSITION 2.25. *For any local system $\underline{A} = \{A_x, \tau_\gamma\}$ on X, the maps τ_γ are isomorphisms with inverses given by $\tau_{\bar{\gamma}}$, where $\bar{\gamma}$ is the inverse path of γ.*

REMARK 2.26. Recall that the category of the fundamental groupoid of a topological space X, denoted by $\Pi(X)$, is defined by having as objects the points of X, as morphisms the homotopy classes of paths relative to the boundary, and with composition defined by the concatenation of paths. Thus by Lemma 2.23 and 2.24 we see that a local system on X is equivalent to a functor from $\Pi(X)$ to **Ab**.

This new viewpoint motivates the following definition:

DEFINITION 2.27. We say that two local systems $\underline{A} = \{A_x, \tau_\gamma\}$ and $\underline{B} = \{B_x, \tau'_\gamma\}$ on X are equivalent if there exists a natural isomorphism from the functor induced by $\underline{A}$ to the induced one by $\underline{B}$.

PROPOSITION 2.28. *If X is a simply connected topological space, then any local system on X is equivalent to a constant local system on X.*

Proof. Fix $x_0 \in X$. We will write $[\gamma]$ for the homotopy type of a path γ in X. Let $\underline{A} = \{A_x, \tau_\gamma\}$ be a local system on X. It corresponds to a functor $F : \Pi(X) \to \mathbf{Ab}$. Define the following constant local system on X by the functor $G : \Pi(X) \to \mathbf{Ab}$ that sends each x to A_{x_0}, and each morphism $[\gamma]$ to the identity map. To prove the assertion, we construct a natural isomorphism $F \Rightarrow G$, i.e. a family $\{\eta_x : A_x \to A_{x_0}\}_{x \in X}$ of abelian group isomorphisms, such that for $[\gamma]$ with $\gamma(0) = x$ and $\gamma(1) = y$ the following commutes:

$$\begin{array}{ccc} A_x & \xrightarrow{\tau_\gamma} & A_y \\ {\scriptstyle \eta_x}\downarrow & & \downarrow{\scriptstyle \eta_y} \\ A_{x_0} & \xrightarrow[\mathrm{id}]{} & A_{x_0}. \end{array}$$

Since X is simply connected, we have that for $x \in X$, the set $\mathrm{Hom}_{\Pi(X)}(x, x_0) = \{[\gamma_x]\}$ consists of a single element. Thus take any representative of $[\gamma_x]$, call it γ_x, and define $\eta_x := \tau_{\gamma_x}$. This map is independent of the choice of γ_x by Lemma 2.23 and an isomorphism by Proposition 2.25. Now given any path γ from x to y, we have that $[\gamma_y * \gamma] = [\gamma_x]$ since each Hom set consists of a single element. So we are able to conclude that $\eta_y \tau_\gamma = \eta_x$ by Lemma 2.24. □

2.3 Homology with Coefficients in a Local System

We now introduce a new homology theory, that in some ways is very similar to singular homology but in others presents certain advantages.

NOTATION 2.29. Let X be a topological space. For any singular n-simplex $\sigma : \Delta^n \to X$, and for any $0 \leq i \leq n$, write $\sigma_i = \sigma(e_i)$, where e_i is the i-th vertex of the simplex. In addition, for any $0 \leq i \leq j \leq n$ define the singular 1-simplex $\sigma_{ij} : \Delta^1 \to X$ to be the composition:

$$\begin{array}{ccccc} \Delta^1 & \longrightarrow & \Delta^n & \xrightarrow{\sigma} & X \\ & \searrow & & \nearrow_{\sigma_{ij}} & \end{array}$$

where the unlabelled map is defined to be the affine map sending e_0 to e_i and e_1 to e_j.

CONSTRUCTION 2.30. Let X be a topological space, and $n \in \mathbb{N}$. Let $\underline{A} = \{A_x, \tau_\gamma\}$ be a local system on X. We want to construct a chain complex using $\underline{A}$. Define the abelian group $C_n(X; \underline{A})$ by:

$$C_n(X; \underline{A}) := \bigoplus_{\sigma : \Delta^n \to X} A_{\sigma_0},$$

and by setting $C_{-1}(X; \underline{A}) = 0$. Next, in order to define the boundary operator, recall the universal property of direct sums of abelian groups: for any direct sum $\bigoplus_{j \in J} A_j$ of abelian groups, given an abelian group B and a group homomorphism $\varphi_k : A_k \to B$, where $k \in J$, there is a unique group homomorphism $\tilde{\varphi} : \bigoplus_{j \in J} A_j \to B$ such that the following commutes for all k:

$$\begin{array}{ccc} A_k & & \\ \downarrow & \searrow^{\varphi_k} & \\ \bigoplus_{j \in J} A_j & \xrightarrow[\tilde{\varphi}]{} & B. \end{array}$$

Given $\sigma : \Delta^n \to X$, we have

$$(\sigma \circ \varepsilon_i)_0 = (\sigma \circ \varepsilon_i)(e_0) = \begin{cases} \sigma_0, & \text{if } i > 0; \\ \sigma_1, & \text{if } i = 0. \end{cases}$$

Thus for $i > 0$, it makes sense to define ∂_n^i to be the unique group homomorphism such that the following commutes:

$$\begin{array}{ccc} A_{\sigma_0} & & \\ \downarrow & \searrow^{\text{id}} & \\ C_n(X; \underline{A}) & \xrightarrow[\partial_n^i]{} & C_{n-1}(X; \underline{A}). \end{array}$$

where the identity map is to be understood as sending A_{σ_0} to $A_{(\sigma \circ \varepsilon_i)_0} \in C_{n-1}(X; \underline{A})$.

If $i = 0$, we have to take our coefficients from $A_{\sigma(e_0)}$ to $A_{\sigma(e_1)}$ since $(\sigma \circ \varepsilon_0)(e_0) = \sigma(e_1)$. For this consider the map $\sigma_{01} : \Delta^1 \to X$. This path has as endpoints $\sigma_{01}(0) = \sigma_0$ and $\sigma_{01}(1) = \sigma_1$. Thus we get an induced map on the local system $\tau_{\sigma_{01}} : A_{\sigma_0} \to A_{\sigma_1} = A_{(\sigma \circ \varepsilon_0)_0}$. Define ∂_n^0 to be the unique map such that the following commutes:

$$\begin{array}{ccc} A_{\sigma_0} & & \\ \downarrow & \searrow & \\ C_n(X;\underline{A}) & \xrightarrow[\partial_n^0]{} & C_{n-1}(X;\underline{A}). \end{array}$$

where the unlabeled map is the composition:

$$A_{\sigma_0} \xrightarrow{\tau_{\sigma_{01}}} A_{(\sigma\circ\varepsilon_0)_0} \hookrightarrow C_{n-1}(X;\underline{A}).$$

Putting all of this together, define the boundary operator for all $n > 0$ as:

$$\partial_n = \sum_{i=0}^{n} (-1)^i \partial_n^i,$$

and set $\partial_0 := 0$. The final step in showing that $C_\bullet(X;\underline{A})$ is a chain complex is to prove that $\partial^2 = 0$ and we refer to [11, p.16] for the gory details.

As usual when presented with a chain complex, we take its homology:

Definition 2.31. The n-th singular homology group $H_n(X;\underline{A})$ of a topological space X with coefficients in a local system $\underline{A}$ is the n-th homology group $H_n(C_\bullet(X;\underline{A}))$ of the complex $C_\bullet(X;\underline{A})$.

Remark 2.32. If $\underline{A}$ is a constant local system with values in an abelian group B, we recover singular homology,i.e. $H_n(X;\underline{A}) = H_n(X;B)$.

2.4 Local System Induced by the Fibres of a Serre Fibration

In the following, we are going to construct a primary ingredient that we need to formulate the LSS, a local system induced by a Serre fibration. One of the main benefits of this system is that it will allow us to encode the "twistedness" of a fibration.

First, we need the following lemma:

Lemma 2.33 ([11, Lem. 2.2.10]). *Suppose that:*

$$\begin{array}{ccc} E' & \xrightarrow{f} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow[g]{} & B. \end{array}$$

is a pullback square in ***Top****, for which $p : E \to B$ is a Serre fibration. If $g : B' \to B$ is a weak equivalence, then so is $f : E' \to E$.*

Construction 2.34. Let $p : E \to B$ be a Serre fibration, and $q \in \mathbb{Z}$ a fixed integer. For each x in B, define the fibre $F_x := p^{-1}(x)$. Define a local system $\underline{A} = \{A_x, \tau_\gamma\}$ as follows. Define $A_x := H_q(F_x)$ for all $x \in B$. Given a path γ, interpret it as a 1-simplex $\gamma : \Delta^1 \to B$ and define $E_\gamma \to \Delta^1$ by the pullback square:

$$\begin{array}{ccc} E_\gamma & \longrightarrow & E \\ {\scriptstyle p_\gamma}\downarrow & & \downarrow{\scriptstyle p} \\ \Delta^1 & \xrightarrow[\gamma]{} & B \end{array}$$

that is to say: $E_\gamma = \Delta^1 \times_B E$. By Proposition 2.14, $p_\gamma : E_\gamma \to \Delta^1$ is a fibration. Furthermore, we can pull this fibration back by the inclusions $\iota_j : \{j\} \to \Delta^1$ with $j = 0, 1$ to obtain the following pullbacks at the fibres:

$$\begin{array}{ccccc} F_{\gamma(0)} & \longrightarrow & E_\gamma & \longleftarrow & F_{\gamma(1)} \\ \downarrow & & {\scriptstyle p_\gamma}\downarrow & & \downarrow \\ \{0\} & \longrightarrow & \Delta^1 & \longleftarrow & \{1\} \end{array}$$

Then as Δ^1 is simply connected, both maps in the bottom row are weak equivalences. Thus by Lemma 2.33 the top rows are also weak equivalences. The weak equivalence axiom of singular homology [14, Ch.16] tells us that weak equivalences induce isomorphisms on the level of homology. Define τ_γ to be the composition

$$H_q(F_{\gamma(0)}) \xrightarrow{\cong} H_q(E_\gamma) \xrightarrow{\cong} H_q(F_{\gamma(1)}).$$

We must now check the axioms of a local system. It is clear that if γ is constant that then τ_γ is the identity.

What remains to be shown is that given a 2-simplex $h : \Delta^2 \to B$, we have:

$$\tau_{h\circ\varepsilon_0}\tau_{h\circ\varepsilon_2} = \tau_{h\circ\varepsilon_1}.$$

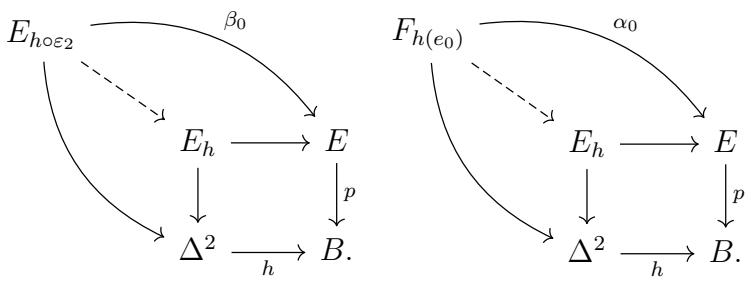
Define E_h as the pullback:

$$\begin{array}{ccc} E_h & \longrightarrow & E \\ \downarrow & & {\scriptstyle p}\downarrow \\ \Delta^2 & \xrightarrow[h]{} & B. \end{array}$$

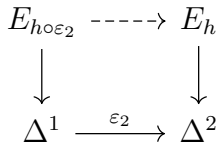
Notice that $F_{h(e_0)}$ and $E_{h\circ\varepsilon_2}$ are obtained as the following pullbacks:

$$\begin{array}{ccc} F_{h(e_0)} & \xrightarrow{\alpha_0} & E \\ \downarrow & & {\scriptstyle p}\downarrow \\ \{h(e_0)\} & \hookrightarrow & B. \end{array} \qquad \begin{array}{ccc} E_{h\circ\varepsilon_2} & \xrightarrow{\beta_0} & E \\ \downarrow & & {\scriptstyle p}\downarrow \\ \Delta^1 & \xrightarrow[h\circ\varepsilon_2]{} & B. \end{array}$$

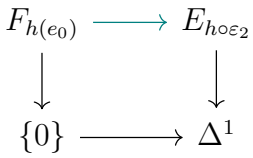
Then by the universal property of pullbacks, we obtain the following induced maps, that we are going to mark by dashed arrows:



By Proposition 2.21, we have that

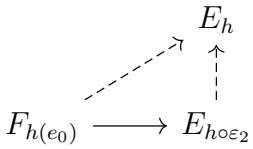


is a pullback square and thus again by Proposition 2.21

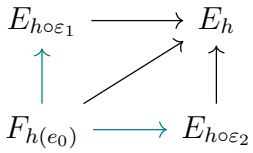


is also a pullback square.

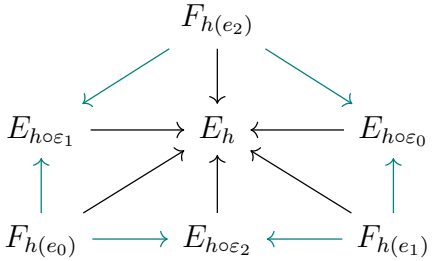
As $h \circ \varepsilon_2$ is a path in B, we know from the above that the map $F_{h(e_0)} \to E_{h\circ\varepsilon_2}$ is a weak equivalence, which we are going to mark by the colour teal in the following. In addition, we have commutativity of the following triangle:



Repeating the same argument, but with $E_{h\circ\varepsilon_1}$ instead of $E_{h\circ\varepsilon_2}$ leads to the commutative diagram:



Using symmetric arguments, we conclude commutativity of the following monstrosity:



To finish the proof, notice that taking homology counter-clockwise along the boundary starting at $F_{h(e_0)}$ and ending at $F_{h(e_2)}$, yields $\tau_{h\circ\varepsilon_0}\tau_{h\circ\varepsilon_2}$. Traversing the diagram clockwise yields $\tau_{h\circ\varepsilon_1}$, and thus the claim follows by commutativity.

Remark 2.35. Since weak equivalences also induce isomorphisms on the level of cohomology, the exact same construction goes through in cohomology and we obtain a local system $\underline{H^q}(F)$ where $\underline{H^q}(F)_x = H^q(F_x)$ and the maps τ are given to us by the fact that all the fibres are weakly equivalent.

2.5 Dress' Construction

The following elegant construction is due to A. Dress (1967). We are going to closely follow the exposition given in [11].

THEOREM 2.36 (Serre spectral sequence [11, Thm. 2.3.1]). *Let $f : E \to B$ be a Serre fibration. Then there is a first quadrant spectral sequence $\{E^r, d^r\}_{r\geq 2}$, called the Serre spectral sequence, with second page $E^2_{p,q} = H_p(B; \underline{H_q}(F))$: The singular homology with coefficients in the local system induced by the fibres of f. The spectral sequence converges to the singular homology of the total space E:*

$$E^2_{p,q} = H_p(B; \underline{H_q}(F)) \Rightarrow H_{p+q}(E).$$

Proof. The idea of the proof is to construct a double complex and then use Theorem 1.18 to obtain two spectral sequences that eventually converge to the same target. One spectral sequence will turn out to be the Serre spectral sequence, and the other will help us prove convergence. First we are going to build the double complex.

For all $p, q \in \mathbb{N}$, consider the set:

$$S_{p,q} := \{(\sigma_{p,q}, \tau_p) \mid \sigma_{p,q} : \Delta^p \times \Delta^q \to E, \tau_p : \Delta^p \to B \text{ continuous with } f\sigma_{p,q} = \tau_p \mathrm{pr}_1\}$$

where $\mathrm{pr}_1 : \Delta^p \times \Delta^q \to \Delta^p$ is the projection onto the first factor. Thus $(\sigma_{p,q}, \tau_p)$ belongs to $S_{p,q}$ if and only if the following commutes:

$$\begin{array}{ccc} \Delta^p \times \Delta^q & \xrightarrow{\sigma_{p,q}} & E \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle f} \\ \Delta^p & \xrightarrow[\tau_p]{} & B. \end{array}$$

Now mirroring the construction of singular chain complexes, apply the free abelian group functor $\mathcal{F}_{Ab}$ (see for example [7] if not acquainted with its definition) and define:

$$K_{p,q} := \mathcal{F}_{Ab}(S_{p,q}) \qquad \forall p, q \geq 0,$$

and set $K_{p,q} := 0$ if $p < 0$ or $q < 0$. All that is left for the construction of a double complex are the horizontal and vertical boundary operators:

$$\begin{aligned} \partial'_{p,q} &: K_{p,q} \to K_{p-1,q} \\ &(\sigma_{p,q}, \tau_p) \mapsto \sum_{i=0}^{p} (-1)^i (\sigma_{p,q}(\varepsilon^p_i \times \mathrm{id}_{\Delta^q}), \tau_p \circ \varepsilon^p_i) \\ \partial''_{p,q} &: K_{p,q} \to K_{p,q-1} \\ &(\sigma_{p,q}, \tau_p) \mapsto \sum_{j=0}^{q} (-1)^{j+p} (\sigma_{p,q}(\mathrm{id}_{\Delta^p} \times \varepsilon^q_j), \tau_p), \end{aligned}$$

and set $\partial'_{p,q} = \partial''_{p,q} = 0$ whenever $p \leq 0$ or $p \leq 0$. Carefully analysing and rearranging terms yields:

$$(\partial')^2 = (\partial'')^2 = \partial''\partial' + \partial'\partial'' = 0.$$

Thus $(K, \partial', \partial'')$ is a double complex. Making use of Theorem 1.18, we obtain two spectral sequences $\{{}^I E^r, {}^I d^r\}$ and $\{{}^{II} E^r, {}^{II} d^r\}$.

1. Let us first examine the second sequence $\{{}^{II}E^r, {}^{II}d^r\}$, and for the sake of readability drop the superscript II in what follows. The first page is given by $E^1_{p,q} = H'_{p,q}(K)$. In order to get a better grasp on what the boundary operator looks like, let us give a different point of view of our construction of the double complex.

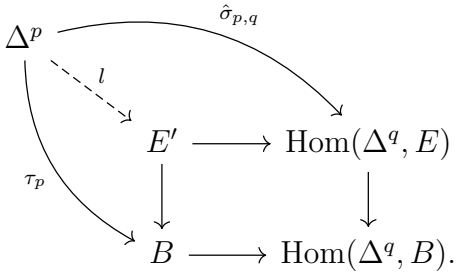
Fix q, and think of p as varying. Since we have a homeomorphism $\Delta^q \cong I^q$, the standard q-simplex is locally compact and Hausdorff. Thus by Proposition 2.11 there is a bijection between continuous maps $\sigma_{p,q} : \Delta^p \times \Delta^q \to E$ and continuous maps $\hat{\sigma}_{p,q} : \Delta^p \to \mathrm{Hom}(\Delta^q, E)$, where $\hat{\sigma}_{p,q}$ is defined by $\hat{\sigma}_{p,q}(t)(s) = \sigma_{p,q}(t, s)$.

Now consider the topological space E' defined by the pullback square:

$$\begin{array}{ccc} E' & \xrightarrow{\pi_1} & \mathrm{Hom}(\Delta^q, E) \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \hat{f}} \\ B & \xrightarrow[\iota]{} & \mathrm{Hom}(\Delta^q, B) \end{array}$$

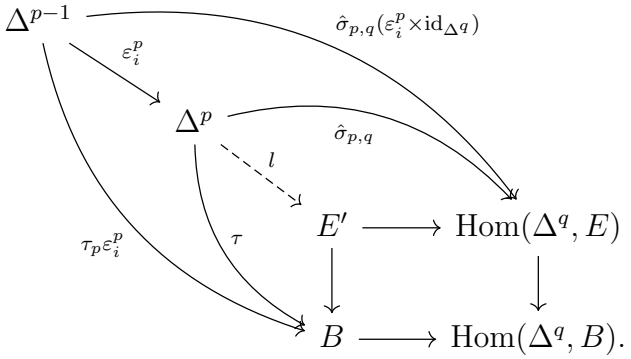
where the map $\iota : B \to \mathrm{Hom}(\Delta^q, B)$ assigns to every point $b \in B$ the constant map $c_b : \Delta^q \to B$ at b. Define the set $P_{p,q}$ to be the set of all continuous maps $\Delta^p \to E'$. We claim that there is a bijection of sets between $P_{p,q}$ and $S_{p,q}$.

For all $(\sigma_{p,q}, \tau_p) \in S_{p.q}$, we have for $s \in \Delta^p$ and $t \in \Delta^q$ that $\hat{f}((\hat{\sigma}_{p,q})(s))(t) = f(\sigma_{p,q}(s, t)) = \iota(\tau_p(s))(t)$. So by the universal property of the pullback, there exists a unique map $l : \Delta^p \to E$ such that the following commutes:



The uniqueness of the map l then insures that the map $F : S_{p,q} \to P_{p,q}$ sending $(\hat{\sigma}_{p,q}, \tau_p)$ to l, is a bijection with obvious inverse given by $F^{-1} : P_{p,q} \to S_{p,q}$, $l \mapsto (\pi_1 l, \pi_2 l)$.

We now want to apply this bijection to the horizontal boundary operator. For this we need to understand how F acts on elements of the form $(\sigma_{p,q}(\varepsilon^p_i \times \mathrm{id}_{\Delta^q}), \tau_p \circ \varepsilon^p_i)$. Consider the commutative diagram:



Thus we see that $F(\sigma_{p,q}(\varepsilon_i^p \times \mathrm{id}_{\Delta^q}), \tau_p \circ \varepsilon_i^p) = l \circ \varepsilon_i^p$. Consequently, the boundary operator takes on the following form under this bijection:

$$\begin{aligned} \partial'_{p,q} : K_{p,q} &\to K_{p-1,q} \\ l &\mapsto \sum_{i=0}^{p} (-1)^i l \circ \varepsilon_i^p. \end{aligned}$$

But this is exactly the boundary operator for the singular chain complex $C_p(E')$. Since $B \to \mathrm{Hom}(\Delta^q, B)$ is a homotopy equivalence by Proposition 2.16, hence in particular a weak equivalence, by Lemma 2.33 the top row $E' \to \mathrm{Hom}(\Delta^q, E)$ is a weak equivalence. Again by Proposition 2.16, the map $E \to \mathrm{Hom}(\Delta^q, E)$ is a weak equivalence. Putting all of this together yields:

$$H_p(E') \xrightarrow{\cong} H_p(\mathrm{Hom}(\Delta^q, E)) \xleftarrow{\cong} H_p(E).$$

And so $H_p(E') \cong H_p(E)$ for all $p \in \mathbb{N}$. Combining this with the above, we have that the homology of the rows, $H'_{p,q}(K)$, is equal to the singular homology groups $H_p(E)$ for each $q \in \mathbb{N}$.

In order to identify the second page $E^2_{p,q}$, recall that

$$E^2_{p,q} = H''_p H'_q(K) := H_p(H'_{q,\bullet}(K), \bar{\partial}'').$$

Applying our previous bijection to the induced vertical maps $\bar{\partial}''$, we get:

$$\begin{aligned} \partial''_{p,q} : K_{p,q} &\to K_{p,q-1} \\ l &\mapsto \sum_{j=0}^{q} (-1)^{j+p} l. \end{aligned}$$

Thus the iterated homology of each row becomes easy to compute, for each fixed column p we get:

$$\ldots \xrightarrow{=} H_p(E) \xrightarrow{0} H_p(E) \xrightarrow{=} H_p(E) \xrightarrow{0} H_p(E) \xrightarrow{=} 0.$$

Therefore, we have

$$E^2_{p,q} = \begin{cases} 0, & \text{if } p > 0; \\ H_q(E), & \text{if } p = 0. \end{cases}$$

Hence we see that further applications of the differential d^r do nothing on the level of homology, and thus our spectral sequence is already stable on the 2nd page , i.e. $E^r_{p,q} = E^2_{p,q}$ for all $r \geq 2$, since $E^{r+1}_{p,1} = H_{p,q}(E^r, d^r)$. So we have

$$E^\infty_{p,q} = E^2_{p,q}.$$

We are now able to compute the homology of the total complex. Recall that double complexes came with filtrations that induce a filtration on the homology of the total complex $H_{p+q}(\mathrm{tot}(K)_\bullet)$:

$$0 = F_{-1}H_{p+q}(\mathrm{tot}(K)_\bullet) \subseteq \cdots \subseteq F_{p+q-1}H_{p+q}(\mathrm{tot}(K)_\bullet) \subseteq F_{p+q}H_{p+q}(\mathrm{tot}(K)_\bullet) = H_{p+q}(\mathrm{tot}(K)_\bullet)$$

such that $E^2_{p,q} = E^\infty_{p,q} \cong \frac{F_p H_{p+q}(\mathrm{tot}(K)_\bullet)}{F_{p-1}H_{p+q}(\mathrm{tot}(K)_\bullet)}$. Fix $n := p+q$ and abbreviate $F_p H_{p+q}(\mathrm{tot}(K)_\bullet)$ by F_p. We have:

$$
\begin{aligned}
H_n(E) = E^2_{0,n} = \frac{F_0}{F_{-1}} = F_0 &\Rightarrow F_0 = H_n(E) \\
0 = E^2_{1,n-1} = \frac{F_1}{F_0} &\Rightarrow F_1 = H_n(E) \\
&\vdots \\
0 = E^2_{n,0} = \frac{F_n}{F_{n-1}} &\Rightarrow F_n = H_n(\mathrm{tot}(K)_\bullet) = H_n(E) = F_{n-1}.
\end{aligned}
$$

We have thus shown that

$$H_{p+q}(\mathrm{tot}(K)_\bullet) = H_{p+q}(E)_\bullet.$$

2. We now analyse the other spectral sequence $\{{}^I E^r_{p,q}, {}^I d^r\}$. In the following, we will again drop the superscript. From Theorem 1.18, we get $E^1_{p,q} = H''_{p,q}(K)$, the homology of the columns. Our goal is to show that $H''_{p,q}(K) \cong C_p(B; H_q(F))$. We take a similar approach as above. Define the set $S_{p,q}(\tau_p) := \{\sigma_{p,q} : \overline{\Delta^p} \times \Delta^q \to E \text{ continuous } \mid f\sigma_{p,q} = \tau_p \mathrm{pr}_1\}$. Note that

$$S_{p,q} = \bigsqcup_{\tau_p : \Delta^p \to B} S_{p,q}(\tau_p).$$

Fix p and a map $\tau_p : \Delta^p \to B$. Consider the fibre of the induced Serre fibration $\hat{f} : \mathrm{Hom}(\Delta^p, E) \to \mathrm{Hom}(\Delta^p, B)$ defined by the pullback square:

$$
\begin{array}{ccc}
F_{\tau_p} & \longrightarrow & \mathrm{Hom}(\Delta^p, E) \\
\downarrow & & \downarrow{\scriptstyle \hat{f}} \\
\{*\} & \xrightarrow[j]{} & \mathrm{Hom}(\Delta^p, B)
\end{array}
$$

where $j(*) = \tau_p$. Define the set $L_{p,q}(\tau_p)$ to be the set of all continuous maps $\Delta^p \to F_{\tau_p}$. As before, one argues that there is a bijection between $L_{p,q}(\tau_p)$ and $S_{p,q}(\tau_p)$ by using the universal property of the pullback.

We now claim that F_{τ_p} and $F_{\tau_p(e_0)}$ are weakly equivalent. Consider:

$$
\begin{array}{ccccc}
F_{\tau_p} & \hookrightarrow & \mathrm{Hom}(\Delta^p, E) & \xrightarrow{\hat{f}} & \mathrm{Hom}(\Delta^p, B) \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\
F_{\tau_p(e_0)} & \hookrightarrow & E & \xrightarrow{\quad f \quad} & B
\end{array}
$$

where as usual the column maps are given by restricting a map to its zero vertex. By Proposition 2.16, the two vertical maps on the right are weak equivalences. As f and $\hat{f}$ are Serre fibrations, by Proposition 2.19 we have the following commutative diagram, where the rows are exact:

$$\begin{array}{ccccccccccccc}
\ldots & \longrightarrow & \pi_{n+1}(\mathrm{Hom}(\Delta^p, B)) & \longrightarrow & \pi_n(F_{\tau_p}) & \longrightarrow & \pi_n(\mathrm{Hom}(\Delta^p, E)) & \longrightarrow & \pi_n(\mathrm{Hom}(\Delta^p, B)) & \longrightarrow & \pi_{n-1}(F_{\tau_p}) & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle \pi_{n+1}(h)} & & \downarrow{\scriptstyle \pi_n(\varphi)} & & \downarrow{\scriptstyle \pi_n(g)} & & \downarrow{\scriptstyle \pi_n(h)} & & \downarrow{\scriptstyle \pi_{n-1}(\varphi)} & & \\
\ldots & \longrightarrow & \pi_{n+1}(B) & \longrightarrow & \pi_n(F_{\tau_p(e_0)}) & \longrightarrow & \pi_n(E) & \longrightarrow & \pi_n(B) & \longrightarrow & \pi_{n-1}(F_{\tau_p(e_0)}) & \longrightarrow & \ldots
\end{array}$$

We also remark that commutativity follows from the functoriality of π_n. Using the 5-Lemma and especially Remark 2.18, we see that $\pi_n(\varphi)$ is an isomorphism for all $n \geq 1$.

For the case $n = 0$, notice that since Δ^p is contractible, there exists a homotopy $G : I \times \Delta^p \to \Delta^p$ such that $G(0, s) = s$ and $G(1, s) = e_0$ for all $s \in \Delta^p$. Define the homotopy $F : I \times \Delta^p \to E$, $F(t, s) := \tau_p(G(t, s))$. This is then a homotopy between $F(0, \bullet) = \tau_p$ and $F(1, \bullet) = c_{\tau_p(e_0)}$.

Since I and Δ^p are locally compact and Hausdorff, by the Exponential Law 2.11 paths $I \to \mathrm{Hom}(\Delta^p, E)$ correspond to maps $I \times \Delta^p \to E$, i.e. homotopies. Thus τ_p and $c_{\tau_p(e_0)}$ are connected by a continuous path. By Proposition 2.20, the fibres F_{τ_p} and $F_{c_{\tau_p(e_0)}}$ are then weakly equivalent.

Now showing that $\pi_0(\varphi) : \pi_0(F_{c_{\tau_p(e_0)}}) \to \pi_0(F_{\tau_p(e_0)})$ is surjective is easy. Let $e \in F_{\tau_p(e_0)}$, then $c_e \in F_{c_{\tau_p(e_0)}}$ and $\mathrm{ev}_0(c_e) = e$.

To show injectivity, we have to show that if $\sigma, \sigma' \in F_{c_{\tau_p(e_0)}}$ are connected by a path γ in $\mathrm{Hom}(\Delta^p, E)$, then $\sigma(e_0)$ and $\sigma'(e_0)$ are also connected by a path. But $\tilde{\gamma}(t) := \gamma(t)(e_0)$ furnishes such a path.

Hence we have shown that F_{τ_p} and $F_{\tau_p(0)}$ are weakly equivalent, thus showing that $H_q(F_{\tau_p}) \cong H_q(F_{\tau_p(0)})$ for all $q \geq 0$.

Computing the homology of the columns:

$$\begin{aligned}
K_{p,q} &= \mathcal{F}_{Ab}(S_{p,q}) \\
&= \mathcal{F}_{Ab}\Big(\bigsqcup_{\tau_p : \Delta^p \to B} S_{p,q}(\tau_p) \Big) \\
&= \bigoplus_{\tau_p : \Delta^p \to B} \mathcal{F}_{Ab}(S_{p,q}(\tau_p)) \\
&= \bigoplus_{\tau_p : \Delta^p \to B} \mathcal{F}_{Ab}(L_{p,q}(\tau_p)).
\end{aligned}$$

For a fixed τ_p, we have:

$$\begin{aligned}
\partial''_{\tau_p} : \mathcal{F}_{Ab}(L_{p,q}(\tau_p)) &\to \mathcal{F}_{Ab}(L_{p,q-1}(\tau_p)) \\
l &\mapsto \sum_{j=0}^{q} (-1)^{j+p} l \circ \varepsilon^q_i,
\end{aligned}$$

which is, possibly up to a sign, the usual boundary operator of the singular chain

complex. Thus

$$\begin{aligned} H''_{p,q} &= \bigoplus_{\tau_p \to B} H_q(\mathcal{F}_{Ab}(L_{p,q}(\tau_p)), \partial''_{\tau_p}) \\ &= \bigoplus_{\tau_p \to B} H_q(F_{\tau p}) \\ &\cong \bigoplus_{\tau_p \to B} H_q(F_{\tau_p(0)} \\ &= C_p(B; \underline{H_q}(F)). \end{aligned}$$

By Theorem 1.18, we have that $E^2_{p,q} = H'_{p,q}H''_q(K) = H_p(C_\bullet(B; \underline{H_q}(F)), \bar{\partial}')$. Thus all that is left to do is to identify $\bar{\partial}'$ in the language of our bijection. Consider:

$$\begin{aligned} \bar{\partial}'_{p,q} : K_{p,q} &\to K_{p-1,q} \\ l &\mapsto \sum_{i=0}^{p} (-1)^i l_i, \end{aligned}$$

where $l_i : \Delta^q \to F_{\tau_p \varepsilon^p_i}$, $l_i(x) := l(x) \circ \varepsilon^p_i \in \mathrm{Hom}(\Delta^{p-1}, E)$.

The final step in identifying $\bar{\partial}'$ is to prove commutativity of the following diagram:

$$\begin{array}{ccc} \bigoplus_{\tau_p : \Delta^p \to B} H_q(F_{\tau_p}) & \xrightarrow{\cong} & \bigoplus_{\tau_p : \Delta^p \to B} H_q(F_{\tau_p(e_0)}) = C_p(B; \underline{H_q}(F)) \\ \Big\downarrow \bar{\partial}' & & \Big\downarrow \delta \\ \bigoplus_{\tau_{p-1} : \Delta^{p-1} \to B} H_q(F_{\tau_{p-1}}) & \xrightarrow{\cong} & \bigoplus_{\tau_{p-1} : \Delta^{p-1} \to B} H_q(F_{\tau_{p-1}(e_0)}) = C_{p-1}(B; \underline{H_q}(F)) \end{array}$$

where δ is the usual boundary map of the complex $C_\bullet(B; \underline{H_q}(F))$. The proof will take us a while. Fix $\tau_p : \Delta^p \to B$. Start with a class $[l]$ in $H_q(F_{\tau_p})$, where $l : \Delta^q \to F_{\tau_p}$ is a continuous map such that $\partial''_{\tau_p}(l) = 0$. The top row isomorphism is the evaluation map that sends $[l]$ to $[\tilde{l}]$ with $\tilde{l} : \Delta^q \to F_{\tau_p(e_0)}$ a continuous map such that $\tilde{l}(s) = l(s)(e_0)$ for all $s \in \Delta^q$. The bottom row isomorphism works analogously. We have that

$$\bar{\partial}'[l] = [\partial' l] = \sum_{i=0}^{p} (-1)^i [l_i].$$

Note that $\tilde{l}_i = \tilde{l}$ for all $i \geq 1$, and $\tilde{l}_0(x) = l(x)(e_1)$ for all $x \in \Delta^q$. Hence the composition:

$$H_q(F_{\tau_p}) \xrightarrow{\bar{\partial}'} \bigoplus_{\tau_{p-1} : \Delta^{p-1} \to B} H_q(F_{\tau_{p-1}}) \cong \bigoplus_{\tau_{p-1} : \Delta^{p-1} \to B} H_q(F_{\tau_{p-1}(e_0)})$$

maps $[l]$ to $[\tilde{l}_0] + \sum_{i=1}^{p} (-1)^i [\tilde{l}]$.

Let us now determine what happens when we traverse the diagram in the other direction. Temporarily add subscripts to the boundary operator δ to make things clearer. The boundary operator can be written as $\delta_p = \sum_{i=0}^{p}(-1)^i \delta_p^i$ where $\delta_p^i([\tilde{l}]) = [\tilde{l}]$ for all $i \geq 1$ and $\delta_p^0([\tilde{l}]) = [\tilde{l}_0]$. To see this, let γ be a path from $\tau_p(e_0)$ to $\tau_p(e_1)$ given by the composition

$$\Delta^1 \hookrightarrow \Delta^p \xrightarrow{\tau_p} B.$$

Write τ_γ for the isomorphism given to us by the local system $\underline{H_q}(F)$. Recall that when we defined τ_γ, we wrote E_γ for the pullback:

$$\begin{array}{ccc} E_\gamma & \longrightarrow & E \\ \downarrow & & \downarrow \\ \Delta^1 & \xrightarrow[\gamma]{} & B \end{array}$$

which induced pullbacks to the fibres:

$$\begin{array}{ccccc} F_{\tau_p(e_0)} & \xrightarrow{\varphi} & E_\gamma & \xleftarrow{\psi} & F_{\tau_p(e_1)} \\ \downarrow & & \downarrow & & \downarrow \\ \{0\} & \hookrightarrow & \Delta^1 & \hookleftarrow & \{1\} \end{array}$$

such that φ and ψ are weak equivalences. The map δ_0^p is given by τ_γ followed by the inclusion into $C_{p-1}(B; \underline{H_q}(F))$. Thus to show that $\delta_p^0([\tilde{l}]) = [\tilde{l}_0]$, we need to show that the following commutes:

$$\begin{array}{ccccc} H_q(F_{\tau_p}) & \xrightarrow{\cong} & H_q(F_{\tau_p(e_0)}) & & \\ \downarrow & & & \searrow^{H_q(\varphi)} & \\ & & & & H_q(E_\gamma) \\ & & & \nearrow_{H_q(\psi)} & \\ H_q(F_{\tau_p \circ \varepsilon_0^p}) & \xrightarrow[\cong]{} & H_q(F_{\tau_p(e_1)}) & & \end{array}$$

where the vertical map sends $[l]$ to $[l_0]$. We now claim that the map $G : F_{\tau_p} \to F_{\tau_p(e_0)} \to E_\gamma$ and $G' : F_{\tau_p} \to F_{\tau_p \circ \varepsilon_0^p} \to F_{\tau_p(e_1)} \to E_\gamma$ are homotopic. For all σ in F_{τ_p}, we have that $G(\sigma) = (0, \sigma(e_0))$ and $G'(\sigma) = (1, \sigma(e_1))$. As usual, define σ_{01} to be the composition $\Delta^1 \hookrightarrow \Delta^p \xrightarrow{\sigma} E$, where the inclusion sends e_0 to e_0 and e_1 to e_1. Define the map H by:

$$\begin{aligned} H : F_{\tau_p} \times I &\to E_\gamma \\ (\sigma, t) &\mapsto (t, \sigma_{01}(t)). \end{aligned}$$

By the definition of F_{τ_p}, we have $f \circ \sigma_{01}(t) = \gamma(t)$ for any t in $I \cong \Delta^1$, and thus $(t, \sigma_{01}(t)) \in E_\gamma$ is well-defined. It is also continuous since it is the composition of continuous functions. Note that $H(\sigma, 0) = G(\sigma)$ and $H(\sigma, 1) = G'(\sigma)$ for all $\sigma \in F_{\tau_p}$. Thus the diagram commutes and we have shown that $\delta_p^0([\tilde{l}]) = [\tilde{l}_0]$.

Hence the composition:

$$H_q(F_{\tau_p}) \cong H_q(F_{\tau_p(e_0)}) \xrightarrow{\delta} \bigoplus_{\tau_{p-1}:\Delta^{p-1}\to B} H_q(F_{\tau_{p-1}(e_0)})$$

maps $[l]$ to $[\tilde{l}_0] + \sum_{i=1}^{p}(-1)^i[\tilde{l}]$. We have thus shown commutativity of the other diagram, and so the induced boundary map $\bar{\partial}'$ is exactly the boundary map of the chain complex $C_\bullet(B; \underline{H_q}(F))$. So we have that $E^2_{p,q} = H_p(B; \underline{H_q}(F))$, which finishes the proof.

□

3 The Serre Spectral Sequence's Greatest Hits

We now put the machinery constructed in the previous sections to work.

3.1 $H^\bullet(SU(n))$ and $H^\bullet(U(n))$

Let $SU(n) := \{A \in U(n) \mid \det(A) = 1\}$. Then in a similar manner to $U(n)$, there exists a Serre fibration

$$SU(n-1) \hookrightarrow SU(n) \to S^{2n-1}.$$

We claim that $H^\bullet(SU(n)) \cong \Lambda(x_3, x_5, \ldots, x_{2n-1})$, the exterior algebra with generator x_i, $\deg x_i = i$. We will prove this by induction. For $n = 2$, the Lie group $SU(2)$ is homeomorphic to S^3, and

$$H^\bullet(SU(n)) \cong H^\bullet(S^3) = \Lambda(x_3).$$

Suppose that $H^\bullet(S^{2n-1}) \cong \Lambda(x_{2n-1})$. The E_2 term of the Serre spectral sequence associated to the above fibration is given by

$$E_2^{p,q} \cong H^p(S^{2n-1}; H^q(SU(n-1))).$$

The Cohomological Universal Coefficients Theorem yields:

$$0 \to H^p(S^{2n-1}) \otimes H^q(SU(n-1)) \to H^p(S^{2n-1}; H^q(SU(n-1))) \to \mathrm{Tor}(H^{p+1}(S^{2n-1}), H^q(SU(n-1))) \to 0$$

where we used that S^{2n-1} is of finite type. Since $H^{p+1}(S^{2n-1})$ is free, the last term vanishes and we are left with

$$\begin{aligned} H^\bullet(S^{2n-1}; H^q(SU(n-1)) &\cong H^\bullet(S^{2n-1}) \otimes H^\bullet(SU(n-1)) \\ &\cong \Lambda(x_{2n-1}) \otimes \Lambda(x_3, x_5, \ldots, x_{2n-3}) = \Lambda(x_3, \ldots, x_{2n-1}). \end{aligned}$$

Thus if $p + q$ is even, we have that $E_2^{p,q}$ is zero. As for all $r \geq 2$, $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ increases the total degree by 1, all differentials are zero and the spectral sequence collapses after the first term. Consequently

$$H^\bullet(SU(n)) \cong \mathrm{Tot}E_\infty^{\bullet,\bullet} \cong \Lambda(x_3, x_5, \ldots, x_{2n-1}).$$

Corollary 3.1. $H^\bullet(U(n)) = \Lambda(x_1, x_3, \ldots, x_{2n-1})$

Proof. We have the fibration $SU(n) \hookrightarrow U(n) \xrightarrow{\det} S^1$ and $U(n) \cong U(1) \times SU(n) \cong S^1 \times SU(n)$. Thus $H^\bullet(U(n)) \cong \Lambda(x_1, \ldots, x_{2n-1})$. □

3.2 The Hurewicz Theorem and Loop Spaces

If a topological space X is path-connected, there exists a link between the first homology and homotopy group:

THEOREM 3.2 (Preliminary Version of the Hurewicz Theorem). *Let X be path-connected. Then we have the following isomorphism:*

$$H_1(X) \cong \pi_1(X)/[\pi_1(X), \pi_1(X)],$$

where the last term is the abelianisation of $\pi_1(X)$.

This classical theorem has a generalisation to higher homotopy groups. Fix $* := (1, 0, \ldots, 0) \in S^n$ as a basepoint.

DEFINITION 3.3. Given a pointed space (X, x_0) and an integer $n \geq 0$, we define the n-th **homotopy group**

$$\pi_n(X, x_0) := [(S^n, *), (X, x_0)] = [S^n, X]_*,$$

the set of pointed maps of S^n into X.

A surprisingly hard proposition to prove is the following:

PROPOSITION 3.4 ([8, Prop. 43.2]). *For every pointed space (X, x_0), $\pi_0(X, x_0)$ is a pointed set, and for every $n \geq 1$, $\pi_n(X, x_0)$ is a group. If $n \geq 2$ then $\pi_n(X)$ is always abelian.*

DEFINITION 3.5. Let (X, x_0) be a pointed space. The **loop space** of (X, x_0) is the pointed space

$$\Omega(X, x_0) := \{u \in \mathrm{Hom}(I, X) \mid u(0) = u(1) = x_0\},$$

where the basepoint is the constant path at x_0. Topologise this set with the compact-open topology. Similarly, define the **path space** PX of (X, x_0) to be the topological space

$$PX(X, x_0) := \{u \in \mathrm{Hom}(I, X) \mid u(0) = x_0\}.$$

Loop spaces are an important ingredient in tackling the hard problem of computing higher homotopy groups.

THEOREM 3.6. *If X is a pointed space and $1 \leq k < n$, one has*

$$\pi_n(X) \cong \pi_{n-k}(\Omega^k X),$$

where Ω^k is the composition of Ω with itself k times.

For every topological space X, we can define a fibration.

PROPOSITION 3.7 ([11, Prop. 3.1.3]). *Let (X, x_0) be a pointed space. The map $p : PX \to B$ defined by $p(u) = u(1)$ for any $u \in PX$ is a fibration, with fibre ΩB. We call it the **path space fibration** of (X, x_0). Furthermore, we have that PX is contractible for every pointed space (X, x_0).*

We are now set up to state the Hurewicz Theorem. If $u : (S^n, *) \to (X, x_0)$ is a pointed map, define $u' : \Delta^n \to X$ to be the same map, precomposed with a homeomorphism $(\Delta^n, \partial\Delta^n) \to (S^n, *)$. We then have a well-defined map

$$\begin{aligned} h : \pi_n(X, x_0) &\to \tilde{H}_n(X) \\ [u] &\mapsto \langle u' \rangle, \end{aligned}$$

where $\tilde{H}_n(X)$ denotes the n-th reduced homology.

THEOREM 3.8 (The Hurewicz Theorem). *Let X be a topological space and let $n \geq 2$. Assume that $\pi_i(X) = 0$ for all $i < n$. Then also $H_i(X) = 0$ for $i < n$ and the Hurewicz map h defined above is an isomorphism $\pi_n(X) \to \tilde{H}_n(X)$.*

Proof by induction: For $n = 2$, we have by assumption that $\pi_1(X) = 0$ and thus the Serre spectral sequence associated to the path fibration $\Omega X \hookrightarrow PX \to X$ has as second page

$$E^2_{p,q} = H_p(X; H_q(\Omega X)).$$

Since ΩX is path-connected as $\pi_0(\Omega X) = \pi_1(X) = 0$, we have that $H_0(\Omega X) = \mathbb{Z}$. This implies that $E^2_{p,0} = H_p(X)$. X is path-connected since $\pi_1(X) = 0$, hence $H_0(X) = \mathbb{Z}$ and $H_0(X; H_q(\Omega X)) = H_q(\Omega X)$. Similarly, $H_1(X) = 0$ as $\pi_1(X) = 0$ and we conclude that $E^2_{0,q} = H_q(\Omega X)$.

As PX is contractible, we have that

$$E^\infty_{p,q} = \begin{cases} \mathbb{Z} \text{ if } p = q = 0, \\ 0 \text{ otherwise.} \end{cases}$$

Since the Serre spectral sequence is a first quadrant spectral sequence, we have that $E^3_{2,0} = E^\infty_{2,0} = 0$ and $E^3_{0,1} = E^\infty_{0,1} = 0$, i.e. the spectral sequence collapses in bidegrees (2,0) and (0,1) after the third term. We have that

$$\begin{aligned} E^3_{0,1} &= \frac{\ker(d^2_{0,1} : H_1(\Omega X) \to 0)}{\operatorname{im}(d^2_{2,0} : H_2(X) \to H_1(\Omega X))} \Rightarrow \operatorname{im} d^2_{2,0} = H_1(\Omega X) \\ E^3_{2,0} &= \frac{\ker(d^2_{2,0} : H_2(X) \to H_1(\Omega X))}{\operatorname{im}(d^2_{4,-1} : 0 \to H_2(X))} \Rightarrow \ker d^2_{2,0} = 0. \end{aligned}$$

It follows that $d^2_{2,0}$ is an isomorphism. On the other hand

$$H_1(\Omega X) \cong \pi_1(\Omega X)/[\pi_1(\Omega X), \pi_1(\Omega X)] \cong \pi_2(X),$$

as $\pi_1(\Omega X) \cong \pi_2(X)$ is abelian. Thus $H_2(X) \cong H_1(\Omega X) \cong \pi_2(X)$.

Assume that the statement is true for $n - 1$. Since $\pi_i(\Omega X) \cong \pi_{i+1}(X)$ we have that we can apply the induction hypotheses to the loop space and

$$h : \pi_{n-1}(\Omega X) \to H_{n-1}(\Omega X)$$

is an isomorphism. By the same argument as above, we have that $d^n_{n,0} : H_n(X) \to H_{n-1}(\Omega X)$ is an isomorphism. □

COROLLARY 3.9. *For $n \geq 2$, we have*

$$\pi_q(S^n) = \begin{cases} 0 & \text{if } q < n \\ \mathbb{Z} & \text{if } q = n. \end{cases}$$

We now use the Serre spectral sequence to compute the homology of the loop space of spheres.

PROPOSITION 3.10 ([10, Thm. 4.33]). *Let $n \geq 2$, then*

$$H_r(\Omega S^n) = \begin{cases} \mathbb{Z} & \text{if } r = k(n-1) \text{ for } k \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Consider the path space fibration $\Omega S^n \hookrightarrow PS^n \to S^n$. Since PS^n is contractible, we have that the limit page of the spectral sequence associated to this particular fibration is equal to

$$E^\infty_{p,q} = \begin{cases} \mathbb{Z} & p = q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Since S^n is simply-connected for $n \geq 2$ and $H_q(S^n) \neq 0$ only for $q = 0$ or $q = n$, we have that the second page is given by

$$E^2_{p,q} = \begin{cases} H_p(S^n; H_q(\Omega S^n)) & p = n \text{ or } p = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Since $d^r : E^r_{p,q} \to E^r_{p-r,p+r-1}$, we must have that for $r < n$, $d_r = 0$. Thus $E^2_{p,q} = E^n_{p,q}$ and the only possible non-zero differential is $d^n : E^n_{n,q} \to E^n_{0,q+n-1}$.

As the Serre spectral sequence is first quadrant, for degree reasons we must have that $d^s_{n,0} = 0$ for all $s > n$. Thus $E^{n+1}_{n,0} = E^\infty_{n,0} = 0$. If a generator $\sigma_{n,0} \in E^n_{n,0} = H_n(S^n)$ is in the kernel of d^n, then it represents a non-zero class in $E^{n+1}_{n,0}$, and thus lives on to the limit page. But $E^\infty_{n,0}$ and thus $\sigma_{n,0}$ cannot lie in the kernel. As d^n is a homomorphism, this means that $k \cdot \sigma_{n,0} \notin \ker d^n$ and so the kernel is empty and

$$\operatorname{im} d^n_{n,0} \cong \mathbb{Z} \subset E^n_{0,n-1} = H_{n-1}(\Omega S^n).$$

To conclude that $H_{n-1}(\Omega S^n) \cong \mathbb{Z}$, we must show that $d^n_{n,0}$ is surjective. If not, there exists $\alpha \in E^n_{0,n-1} \setminus \operatorname{im} d^n_{n,0}$. This means that α is a non-zero class in $E^{n+1}_{0,n-1} = E^\infty_{0,n-1}$, but the latter is zero and thus such an α cannot exist. Hence $H_{n-1}(\Omega S^n) \cong \mathbb{Z}$. Furthermore,

$$E^2_{n,n-1} = H_n(S^n; H_{n-1}(\Omega S^n)) = \mathbb{Z}.$$

Repeating the above argument, we have that $E^2_{n,n-1} = E^n_{n,n-1}$ and

$$d^n : E^n_{n,n-1} \to E^n_{0,2(n-1)} = H_{2(n-1)}(\Omega S^n)$$

is an isomorphism. Thus

$$\mathbb{Z} = E^2_{0,2(n-1)} = H_{2(n-1)}(\Omega S^n).$$

In a similar vein, one obtains $\mathbb{Z} = H_{k(n-1)}(\Omega S^n)$ for all $k \in \mathbb{Z}$.

To conclude the proof, we must show that if $j \neq k(n-1)$ for some $k \in \mathbb{Z}$, that then $H_j(\Omega S^n) = 0$. By way of contradiction, assume $j > 0$ is the smallest non-multiple of $n-1$ such that $E^2_{0,j} = H_j(\Omega S^n) \neq 0$.

For degree reasons, this group cannot be in the image of any differential, as $E^r_{p,q}$ can only be non-zero with $r > 0$ if $p = n$. Again since $E^\infty = 0$ in positive degrees, we must have that for an element $\alpha \in E^2_{0,j}$ to exist, it must lie in $\mathrm{im}(d^n : E^n_{n,s} \to E^n_{0,s+n-1})$ from which we conclude that $j = s + n - 1$. For $E^n_{n,s}$ to be non-zero, we must have that $s = k(n-1)$ since $s < j$ and j was chosen to be minimal. But then $j = s + n - 1 = (k+1)(n-1)$ which is a contradiction. □

3.3 The Gysin and the Wang Sequence

If the fibre of a fibration is a sphere, we are able to construct a long exact sequence.

THEOREM 3.11 (Gysin Sequence, [11, Thm. 3.3.3]). *Let $S^n \hookrightarrow E \to B$ be a Serre fibration, with B simply-connected and $n \geq 1$. There exists an exact sequence:*

$$\cdots \to H_r(E) \to H_r(B) \to H_{r-n-1}(B) \to H_{r-1}(E) \to \dots$$

In particular, for $0 \leq r \leq n-1$ we have the isomorphism $H_r(E) \cong H_r(B)$.

Proof. The second page of the Serre spectral sequence associated to this fibration looks like:

$$E^2_{p,q} = H_p(B; H_q(S^n)) = \begin{cases} H_p(B;\mathbb{Z}) \text{ if } p = 0 \text{ or } p = n, \\ 0 \qquad\qquad \text{otherwise.} \end{cases}$$

The only non-zero differentials are $d^{n+1}_{p,0} : E^{n+1}_{p,0} \to E^{n+1}_{p-n-1,n}$ and $E^{n+1}_{p,q} = E^2_{p,q}$. Since we have a first quadrant sequence, by degree reasons it follows that:

$$E^{n+2}_{p,q} = \cdots = E^\infty_{p,q} = \begin{cases} 0 & \text{if } q \neq 0,\ q \neq n, \\ \ker d^{n+1}_{p,0} & \text{if } q = 0, \\ \mathrm{coker}\, d^{n+1}_{p+n+1,0} & \text{if } q = n. \end{cases}$$

We have the exact sequence

$$0 \to \ker d^{n+1}_{p,0} \to E^{n+1}_{p,0} \xrightarrow{d^{n+1}_{p,0}} E^{n+1}_{p-n-1,n} \to \mathrm{coker}\, d^{n+1}_{p,0} \to 0,$$

which after the above identification becomes

$$0 \to E^\infty_{p,0} \to E^2_{p,0} \xrightarrow{d^{n+1}_{p,0}} E^2_{p-n-1,n} \to E^\infty_{p-n-1,n} \to 0.$$

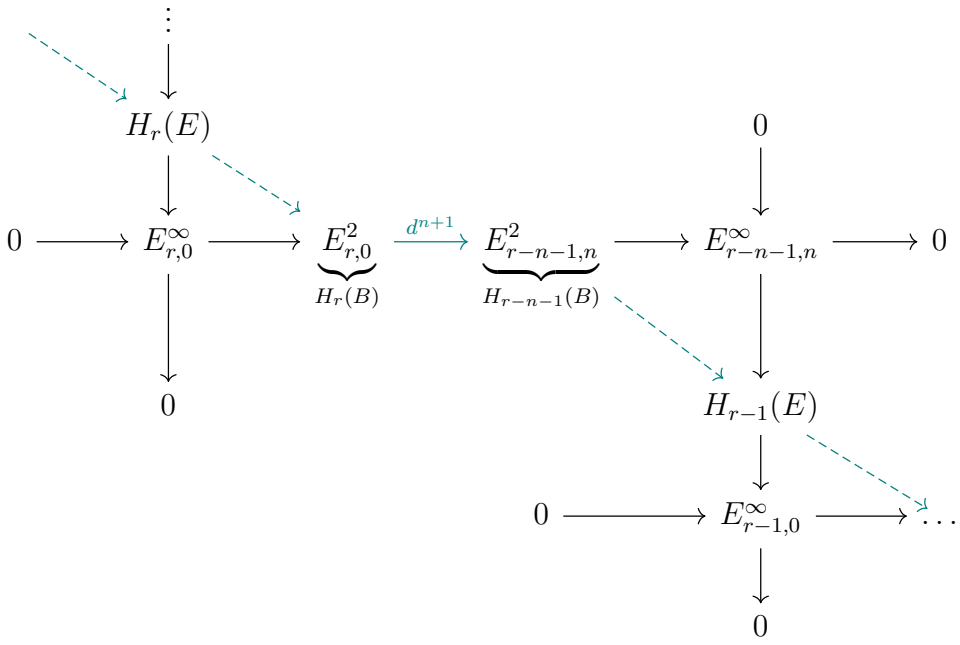
By the structure of the limit page, the filtration of $H_r(E)$ is given by

$$\{0\} = F_0 = \cdots = F_{r-n-1} \subset F_{r-n} = E^\infty_{r-n,n} = \cdots = F_{r-1} \subset F_r = H_r(E).$$

Thus we have the short exact sequence

$$0 \to E^\infty_{r-n,n} \to H_r(E) \to H_r(E)/E^\infty_{r-n,n} = E^\infty_{r,0} \to 0.$$

We gather everything in the following commutative diagram



where the dashed-arrows are given by composing the other two morphisms in their respective triangles. From this we can read off the long exact sequence

$$\dots \longrightarrow H_r(E) \dashrightarrow H_r(B) \xrightarrow{d^{n+1}} H_{r-n-1}(B) \dashrightarrow H_{r-1}(E) \longrightarrow \dots$$

The last statement follows from the fact that if $0 \leq r \leq n-1$, then $r-n-1 \leq 0$ and thus $H_{r-n-1}(B) = 0$. □

A similar result holds when the base space is a sphere. This can also be proved using spectral sequences.

THEOREM 3.12 (Wang Sequence, [11, Thm. 3.3.6]). *Let $F \hookrightarrow E \to S^n$ be a Serre fibration, with $n \geq 2$. Then there exists an exact sequence:*

$$\dots \to H_r(F) \to H_r(E) \to H_{r-n}(F) \to H_{r-1}(F) \to \dots$$

In particular, for $0 \leq r \leq n-2$, we have the isomorphism $H_r(E) \cong H_r(F)$.

Chapter II

The Serre Spectral Sequence in Morse Cohomology

Morse theory is the study of the topology of a smooth manifold B through a special class of smooth functions $f : B \to \mathbb{R}$, the so-called Morse functions. By the analysis of their non-degenerate critical points, these functions are able to compute topological invariants of the manifold. If the number of critical points of a manifold is finite, classical Morse theory is, for example, able to describe the Euler characteristic and the Betti numbers solely by the number of critical points of a certain type. These results are known as the Morse inequalities.

A more modern approach, that later on served as inspiration for the development of Floer homology, is the study of trajectories that link two critical points. These trajectories are special in nature, as they are given by the flow of the (negative pseudo-)gradient vector field of a Morse function with respect to a given metric. By counting these trajectories in a specific fashion, one is then able to construct a chain complex that induces the topological invariant of Morse (co)homology.

Given a (smooth) fibre bundle $F \hookrightarrow E \xrightarrow{\pi} B$, the following chapter shows how a Morse function on the base space B and on the fibre F induce a Morse function on the total space E. We then use this Morse function on E to give a description of the Serre Spectral Sequence purely in terms of its critical points of the involved Morse functions. But first we must show that under certain conditions, Morse and Cellular cohomology agree. The primary reference for this chapter will be [9], which we will follow closely.

4 Identification of Morse and Singular Cohomology

We quickly recap the construction of cellular cohomology and adapt it to the case of local coefficients. We then show that we can use the gradient flow lines of a Morse function to construct a cell complex that, under certain (transversality) conditions on the pseudo-gradient, is chain equivalent to the one of cellular homology.

4.1 Cellular Cohomology with Values in a Local System

Recall our construction of homology with local coefficients section §2.3. We can extend our definition of local systems to include systems $\underline{A} = \{A_x.\tau_\gamma\}$ where the A_x are R-modules for a ring R, and the τ_γ are R-module homomorphisms.

Let $\underline{A} = \{A_x.\tau_\gamma\}$ be a local system (of R-modules). An equivalent definition of p-chains with local coefficients is the following:

$$C_p(B;\underline{A}) := \left\{ \sum_{i=1}^{k} m_i \otimes \sigma_i \mid \sigma_i : \Delta^p \to B, m_i \in A_{\sigma_i(e_0)} \right\}$$

with differential:

$$\partial : C_p(B;\underline{A}) \to C_{p-1}(B;\underline{A}) \qquad \text{such that}$$
$$\partial(m \otimes \sigma) = \tau_{\sigma_{01}}(m) \otimes \partial_0\sigma + \sum_{i=1}^{p} m \otimes \partial_i\sigma.$$

The p-cochains with coefficients in $\underline{A}$ are then defined:

$$C^p(B;\underline{A}) = \mathrm{Hom}(C_p(B;\underline{A}), R)$$

with dual differential $\delta(f) = f \circ \partial$ for $f \in C^p(B;\underline{A})$. We then define:

$$H^p(B;\underline{A}) := H^p(C^\bullet(B;\underline{A}), \delta).$$

We now give a quick introduction to the construction of cellular cohomology with values in a local system. Assume that B admits the structure of a CW-complex with skeleton filtration $\emptyset = B^{-1} \subset B^0 \subset B^1 \subset \cdots \subset B$. Define the chain complex:

$$\mathrm{Cell}^p(B;\underline{A}) := H^p(B^p, B^{p-1};\underline{A})$$
$$\delta_{\mathrm{Cell}} = \delta : H^p(B^p, B^{p-1};\underline{A}) \to H^{p+1}(B^{p+1}, B^p;\underline{A})$$

with δ the connecting homomorphism of the long exact sequence in cohomology associated to the triple (B^{p+1}, B^p, B^{p-1}). The following result shows that this new complex is chain equivalent to the chain complex $C^p(B;\underline{A})$:

THEOREM 4.1. *There is a canonical isomorphism, i.e. only dependent on the CW-decomposition:*

$$H^p(\mathrm{Cell}^\bullet(B;\underline{A}), \delta_{\mathrm{Cell}}) \cong H^p(B;\underline{A}) \qquad p \geq 0.$$

A proof for singular homology that stays valid for coefficients in a local system can for instance be found in [7, Thm. 20.5].

There is a more explicit definition of the cellular boundary operator. Let $(e^p_\alpha)_\alpha$ be the open p-cells and let $x_\alpha \in e^p_\alpha$ be their centres. Suppose further that we have chosen orientations for the e^p_α's, i.e. generators of $H^p(e^p_\alpha, \partial e^p_\alpha;\underline{A}) \cong A_{x_\alpha}$. The following discussion will depend on this choice. We have:

$$\mathrm{Cell}^p(B;\underline{A}) \cong \bigoplus_\alpha A_{x_\alpha}.$$

This follows from the fact that $H^p(B^p, B^{p-1}; \underline{A}) \cong H^p(B^p/B^{p-1}; \underline{A})$ for $p \geq 1$ and B^p/B^{p-1} is equal to a wedge sum of p-spheres, one for each p-cell. In addition $H^0(B^0, \emptyset; \underline{A}) = H^0(B^0)$ for B^0 a finite collection of points.

Given a $(p+1)$-cell e_β^{p+1} and a p-cell e_α^p define the following maps:

$$f_\beta : \partial e_\beta^{p+1} \to B^p$$

the attaching map of e_β^{p+1} and

$$\rho : B^p \to B^p/B^{p-1}$$

the quotient map. Note that again B^p/B^{p-1} is a wedge sum of p-spheres, one for each p-cell. Denote by:

$$q_\alpha : B^p/B^{p-1} \to S_\alpha^p$$

the quotient map that collapses all the p-spheres not corresponding to the cell e_α^p to a point. Define the map

$$h_{\alpha,\beta} : S_\beta^p \to S_\alpha^p$$

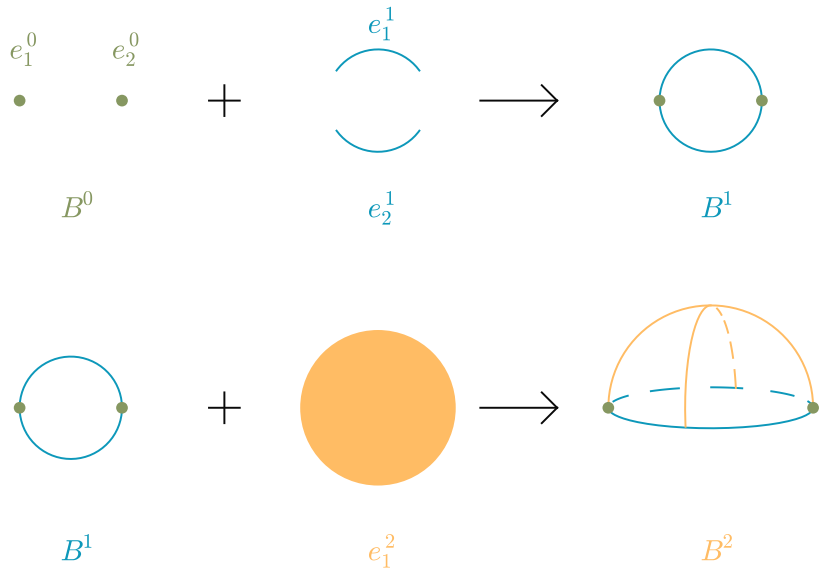
to be the composition

$$\partial e_\beta^{p+1} \xrightarrow{f_\beta} B^p \xrightarrow{\rho} B^p/B^{p-1} \xrightarrow{q_\alpha} S_\alpha^p.$$

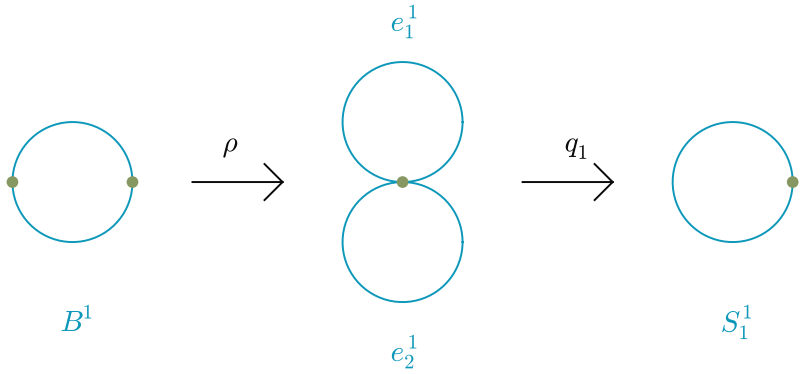
Define the incidence number by

$$[e_\alpha^p : e_\beta^{p+1}] = \deg(h_{\alpha,\beta}).$$

Example 4.2. The closed half-sphere can be constructed as the following cell decomposition:



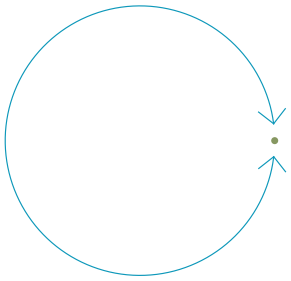
And the collapsing map is given by



In analogy to cellular homology with $\mathbb{Z}$ coefficients, we then define the boundary operator as:

$$\delta_{\text{Cell}}(m_\alpha) = \sum_\beta [e^p_\alpha : e^{p+1}_\beta] \tau_{\alpha\beta}(m_\alpha), \qquad m_\alpha \in A_{x_\alpha}, \tag{II.1}$$

with $\tau_{\alpha\beta}$ being the transport of coefficients in our local system along a path from x_α to x_β contained in $\overline{e^{p+1}_\beta}$ (such a path only doesn't exist if e^p_α does not attach to e^{p+1}_β but in this case the incidence number is 0). In order for this transport to be independent of the choice of path, it is sufficient to assume that $\overline{e^{p+1}_\beta}$ is simply-connected. This only causes problems in dimension 1, where the ends might get identified to form something homeomorphic to S^1.



In this case, let $\tau^{\pm}_{\alpha\beta}$ be paths from x_α to x_β that traverse e^1_β in the positive/negative sense. The term $[e^0_\alpha : e^1_\beta]\tau_{\alpha\beta}(m_\alpha)$ is then zero and needs to be replaced by

$$\langle e^0_\alpha \rangle (\tau^+_{\alpha\beta}(m_\alpha) - \tau^-_{\alpha\beta}(m_\alpha))$$

with $\langle e^0_\alpha \rangle \in \{\pm 1\}$ being the orientation of e^0_α.

Another way of constructing the Serre Spectral sequence is by exact couples.

Construction 4.3 (Serre Exact Couple). Let $E \xrightarrow{\pi} B$ be a smooth fibre bundle with fibre F. Assume further that F and B are compact. Morse theory then tells us that B admits the structure of a finite CW-complex. Such a CW-structure can be interpreted as a filtration:

$$\emptyset \subset B^0 \subset \cdots \subset B^{n-1} \subset B^n = B$$

with B^0 a finite collection of points and B^p with $p \geq 1$ obtained from B^{p-1} by attaching finitely many closed balls of dimension p along their respective boundaries. We call B^p the skeleton of dimension p or p-skeleton. From this filtration, we construct a filtration of E, define:

$$E^p := \pi^{-1}(B^p), \qquad 0 \leq p \leq .n$$

Then

$$\emptyset \subset E^0 \subset \cdots \subset E^{n-1} \subset E^n = E$$

and a filtration of $C = C^\bullet(E)$:

$$0 \subset C_n \subset \cdots \subset C_1 \subset C_0 = C$$

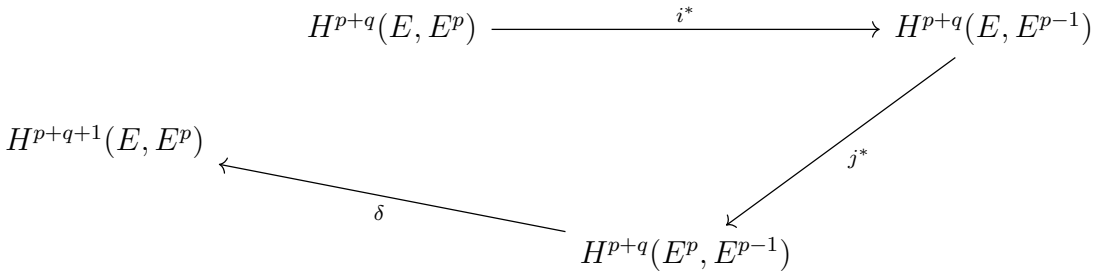
where

$$C_p := \ker(C^\bullet(E) \to C^\bullet(E^{p-1})).$$

The fact that $(E^p, E^{p-1}) \hookrightarrow (E, E^{p-1}) \hookrightarrow (E, E^p)$ provides us with a long exact sequence in cohomology:

$$\ldots \longrightarrow H^{p+q-1}(E^p, E^{p-1}) \xrightarrow{\delta} H^{p+q}(E, E^p) \xrightarrow{i^{p+q}} H^{p+q}(E, E^{p-1}) \xrightarrow{j^{p+q}} H^{p+q}(E^p, E^{p-1}) \xrightarrow{\delta} \ldots$$

and we arrive at the following exact couple:



Define $D_1^{p,q} := H^{p+q}(E, E^{p-1})$, $E_1^{p,q} := H^{p+q}(E^p, E^{p-1})$. It turns out that this exact couple induces the Serre Spectral Sequence, see for example [3, Thm. 5.2].

A version of the Serre Spectral Theorem adapted to the current setting states:

THEOREM 4.4 ([3, § 5.1]). *Let $F \hookrightarrow E \xrightarrow{\pi} B$ be a fibre bundle. Suppose that B admits the structure of a finite CW-complex. The spectral sequence associated to the exact couple of Construction 4.3 then verifies the following properties:*

$$E_1^{p,q} \cong \mathrm{Cell}^p(B; \underline{H}^q(F)), \quad d_1 = \delta_{\mathrm{Cell}};$$
$$E_2^{p,q} \cong H^p(B; \underline{H}^q(F));$$
$$E_r^{p,q} \Rightarrow H^{p+q}(E).$$

The first isomorphism depends on the cellular decomposition of B, the second one does not.

REMARK 4.5. If $K \subset B$ is a cellular sub-complex, we can define the relative cellular homology $H^p_{\mathrm{Cell}}(B, K; \underline{A})$ as the homology of the complex

$$\mathrm{Cell}^p(B, K; \underline{A}) = H^p(B^p, B^{p-1} \cup K^p; \underline{A}),$$

$$H^p(B^p, B^{p-1} \cup K^p) \xrightarrow{i^*} H^p(B^p, B^{p-1}) \xrightarrow{\delta} H^{p+1}(B^{p+1}, B^p) \xrightarrow{\pi} H^{p+1}(B^{p+1}, B^p \cup K^{p+1})$$

$$\delta_{\mathrm{Cell}} = \pi \circ \delta \circ i^*,$$

where π is the projection given by the canonical decomposition

$$H^{p+1}(B^{p+1}, B^p) \cong H^{p+1}(B^{p+1}, B^p \cup K^{p+1}) \oplus H^{p+1}(K^{p+1}, K^p).$$

The analogue of Theorem 4.1 then still holds.

4.2 The Thom-Smale-Witten Complex of a Compact Manifold

Consider a closed manifold B, a Morse function $f : B \to \mathbb{R}$ and a vector field X satisfying:

$$X(f) < 0$$

away from the critical points of f. One calls such a vector field a negative pseudo-gradient with respect to f. Denote the flow of X by Φ_t.

Recall that the stable and unstable manifolds of X are defined as follows:

$$W^s(x) := \{z \in B \mid \lim_{t\to+\infty} \Phi_t(z) = x\} \qquad x \in \mathrm{Crit}(f)$$
$$W^u(x) := \{z \in B \mid \lim_{t\to-\infty} \Phi_t(z) = x\}.$$

By the Stable/Unstable Manifold Theorem for a Morse function [13, Thm. 4.2] these two manifolds are diffeomorphic to open balls. Thus in particular, $W^u(x)$ is contractible. If the Morse index of x with respect to f is equal to k, then $\dim W^u(x) = k$ and $\dim W^s(x) = \dim B - k$.

Definition 4.6. A vector field X satisfies the Morse-Smale condition if every stable manifold is transverse to every unstable manifold:

$$W^u(x) \pitchfork W^s(x) \qquad x, y \in \mathrm{Crit}(f).$$

For a fixed Morse function, the condition of being Morse Smale is generically satisfied in the space of all negative pseudo-gradient vector fields. Define the trajectory space linking two critical points x and y by:

$$\widetilde{\mathcal{M}}(x, y) := \{\gamma : \mathbb{R} \to B \mid \dot\gamma = X(\gamma),\ \lim_{t\to-\infty} \gamma(t) = x,\ \lim_{t\to+\infty} \gamma(t) = y\}.$$

As every path γ is uniquely determined by $\gamma(0)$, we can identify $\widetilde{\mathcal{M}}(x, y)$ and $W^u(x) \cap W^s(y)$. Under the Morse-Smale condition, the latter is a manifold of dimension $\operatorname{ind} x - \operatorname{ind} y$ by transversality. The group $(\mathbb{R}, +)$ acts on $\widetilde{\mathcal{M}}(x, y)$ by translation, $s \cdot \gamma(t) = \gamma(t + s)$ for $s \in \mathbb{R}$, $\gamma \in \widetilde{\mathcal{M}}(x, y)$. The quotient $\widetilde{\mathcal{M}}(x, y)/\mathbb{R}$ is going to be denoted by $\mathcal{M}(x, y)$. We have

$$\dim \mathcal{M}(x, y) = \operatorname{ind} x - \operatorname{ind} y - 1.$$

If $\operatorname{ind} x - \operatorname{ind} y = 1$, we will see in the following that $\mathcal{M}(x, y)$ is a closed manifold of dimension 0, i.e. a finite collection of points. In this case, an orientation $\langle x\rangle$ of the unstable manifolds $W^u(x)$, $x \in \mathrm{Crit}(f)$ induces an assignment of ± 1 to each path γ by the subsequent convention.

Let $z := \gamma(0)$. As $W^u(x)$ is contractible, every vector bundle over $W^u(x)$ is trivial and so $\langle x\rangle$ induces an orientation of the tangent bundle $TW^u(x)$. By transversality again, $\langle y\rangle$ induces an orientation of $T_yW^u(y) \cong T_yB/T_yW^s(y)$.

Since $W^s(y)$ is also contractible, this induces an orientation of the normal bundle $TB/TW^s(y)$ over $W^s(y)$. For the point z, the transversality condition implies the isomorphism:

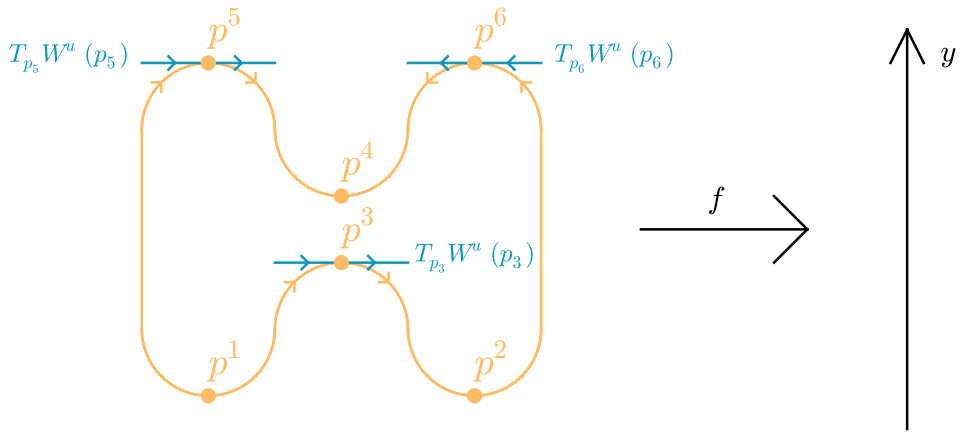
$$T_zW^u(x)/(T_zW^u(x) \cap T_zW^s(y)) \cong T_zB/T_zW^s(y)$$

which entails an orientation of $T_zW^u(x)/(T_zW^u(x) \cap T_zW^s(y))$. As all of the above isomorphisms are canonical, the resulting orientation is only dependent on the choice of $\langle y \rangle$.

Since we have $T_zW^u(x) = T_zW^u(x)/T_z(W^u(x) \cap W^s(y)) \oplus T_z(W^u(x) \cap W^s(y))$ we are able to define an orientation on $T_z(W^u(x) \cap W^s(y))$ by declaring a basis to be positively oriented if by adjoining it to a positive basis of $T_zW^u(x)/T_z(W^u(x) \cap W^s(y))$, we get a positive basis of $T_zW^u(x)$.

Now assume $|x| = |y| + 1$. The vector space $T_zW^u(x)/T_z(W^u(x) \cap W^s(y))$ is of dimension 1 and we associate to γ a sign ± 1 depending on whether $\dot{\gamma}(z)$ is a positive or negative basis.

Example 4.7 ([13, Ex. 7.8]). Consider $M = S^1$ and the Morse-Smale function $f : M \to \mathbb{R}$ given by the height function in the following picture:



We chose to orient $T_{p_5}W^u(p_5)$ and $T_{p_3}W^u(p_3)$ from left to right and $T_{p_6}W^u(p_6)$ from right to left. It is important to note that the arrows drawn on the circle don't refer to the gradient flow lines, but to the choice of orientation of $T_z(W^u(x) \cap W^s(y))$. We denote by $n(p_i, p_j)$ the sign that gets picked up by a path in $\mathcal{M}(p_i, p_j)$ and end up with the following table of signs:

	p_1	p_2	p_3	p_4	p_5	p_6
p_1	0	0	0	0	0	0
p_2	0	0	0	0	0	0
p_3	-1	1	0	0	0	0
p_4	0	0	0	0	0	0
p_5	-1	0	0	1	0	0
p_6	0	-1	0	1	0	0

The TSW complex is normally constructed on $\mathbb{Z}$, but we are going to extend its definition to accommodate for an eventual system of local coefficients $\underline{A}$ on B.

Define

$$C^q_{\text{Morse}}(B, X; \underline{A}) := \bigoplus_{\text{ind}\, x = q} A_x$$

and consider the differential

$$\begin{aligned} \delta_{\text{Morse}} : C^q_{\text{Morse}}(B, X; \underline{A}) &\to C^{q+1}_{\text{Morse}}(B, X; \underline{A}) \\ \delta_{\text{Morse}}(m) &= \sum_{\text{ind}\, y = q+1} \sum_{\gamma \in \mathcal{M}(y,x)} n_\gamma \tau_\gamma^{-1}(m). \end{aligned} \tag{II.2}$$

In order to prove that we have indeed constructed a chain complex, we have to show that $\delta^2_{\text{Morse}} = 0$. We are going to need the following analytical result:

THEOREM 4.8 ([9, Thm. 3.2.13]). *Let $x, y \in \text{Crit}(f)$, $\text{ind}\, x > \text{ind}\, y$ and $(\gamma_j)_{j\geq 1}$ a sequence in $\mathcal{M}(x, y)$. There exists a subsequence $(\gamma'_j) \subset (\gamma)$, together with points $x_1, \ldots, x_k \in \text{Crit}(f)$ with $\text{ind}\, x > \text{ind}\, x_1 > \cdots > \text{ind}\, x_k > \text{ind}\, y$, paths $\gamma^i \in \mathcal{M}(x_i, x_{i+1})$, $x_0 = x$, $x_{k+1} = y$ and reparametrisation times $(s^i_j)_{j\geq 1} \subset \mathbb{R}$ such that:*

$$s^i_j \cdot \gamma'_j \xrightarrow[j\to\infty]{C^\infty_{loc}} \gamma^i \qquad 0 \leq i \leq k.$$

Here C^∞_{loc}-convergence of a family $(f_i : \mathbb{R} \to B)_{i\in\mathbb{N}}$ to a function f is defined by $f_i \xrightarrow{C^\infty} f$ on each compact $K \subset \mathbb{R}$. We sometimes say that $\mathcal{M}(x, y)$ is compact up to broken trajectories.

If $\text{ind}\, x - \text{ind}\, y = 1$, then since the index is an integer, no critical point fits "in between" x and y in the statement of Theorem 4.8. Thus a sequence of trajectories $(\gamma_j)_{j\geq 1} \subset \mathcal{M}(x, y)$ converges to a (non-broken) trajectory $\gamma \in \mathcal{M}(x, y)$, in this case showing compactness of $\mathcal{M}(x, y)$.

When $\text{ind}\, x - \text{ind}\, y = 2$, we are able to visualise this convergence:

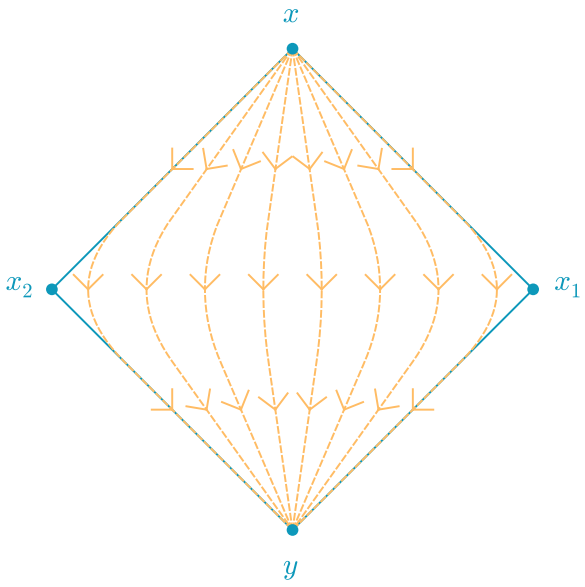


Figure II.1: Once trajectories reach the boundary, they break up into two pieces that flow over the critical points x_1 or x_2 respectively.

The following Theorem ensures that every broken trajectory from x to y can be approximated by sequences in $\mathcal{M}(x, y)$.

THEOREM 4.9 ([9, Thm. 3.2.14]). *Let $x, y, z \in \mathrm{Crit}(f)$, $\operatorname{ind} x > \operatorname{ind} y > \operatorname{ind} z$ and let $(\gamma', \gamma'') \in \mathcal{M}(x, y) \times \mathcal{M}(y, z)$. Then there exists a continuous family $(\gamma_R)_{R>>1} \subset \mathcal{M}(x, z)$ that converges for $R \to \infty$ to the broken trajectory (γ', γ'') and such that for every sequence $(\gamma_j)_j \subset \mathcal{M}(x, z)$ that converges to (γ', γ''), the γ_j are contained in the connected components of γ_R for j big enough.*

To complete the proof of the identity $\delta^2_{\mathrm{Morse}} = 0$, we need the following proposition:

PROPOSITION 4.10 ([4, Prop. 2.4.1]). *Let $x, z \in \mathrm{Crit}(f)$, such that $\operatorname{ind} x = \operatorname{ind} z + 2$. Assume that there is a connected component of $\mathcal{M}(x, z)$ whose ends converge to (γ_1', γ_1'') and (γ_2', γ_2'') with $\gamma_i' \in \mathcal{M}(x, y_i)$, $\gamma_i'' \in \mathcal{M}(y_i, z)$ and $\operatorname{ind} y_i = \operatorname{ind} z + 1$. Then:*

$$n_{\gamma_1'} \cdot n_{\gamma_1''} + n_{\gamma_2'} \cdot n_{\gamma_2''} = 0.$$

These connected components will be called the non compact components.

When $\operatorname{ind} x = \operatorname{ind} z + 2$, we have that $\mathcal{M}(x,z)$ is a one-dimensional manifold, i.e. consists of a union of circles and open intervals. Non compact components in the above setting must thus be diffeomorphic to open intervals, that possess two "ends". The content of the above proposition is then that the orientations of the two ends, induced by the orientations of the unstable manifolds, are opposite. Moreover, these components are simply-connected.

COROLLARY 4.11. *Let $\underline{A}$ be a local system on B. Then:*

$$\delta^2_{\text{Morse}} = 0.$$

Proof. It is enough to show that $\delta^2_{\text{Morse}}(m) = 0$, $m \in A_z$, $\operatorname{ind} z = q$. We have

$$\delta^2_{\text{Morse}}(m) = \sum_{\operatorname{ind} x = q+2} \left(\sum_{\operatorname{ind} y = q+1} \sum_{(\gamma',\gamma'') \in \mathcal{M}(x,y)\times\mathcal{M}(y,z)} n_{\gamma'} n_{\gamma''} \tau_{\gamma'}^{-1}(\tau_{\gamma''}^{-1}(m)) \right). \tag{II.3}$$

By Theorem 4.9 each broken trajectory (γ'_1, γ''_1) is part of the boundary of a connected component with boundary of $\mathcal{M}(x,z)$, and thus the other end (γ'_2, γ''_2) of the component must also appear in the above sum. Since each component with boundary is simply-connected, the paths $\gamma'_1 * \gamma''_1$ and $\gamma'_2 * \gamma''_2$ are homotopic, and so $\tau^{-1}_{\gamma'_1} \circ \tau^{-1}_{\gamma''_1} = \tau^{-1}_{\gamma'_2} \circ \tau^{-1}_{\gamma''_2}$. Furthermore, by Proposition 4.10,

$$n_{\gamma'_1} \cdot n_{\gamma''_1} + n_{\gamma'_2} \cdot n_{\gamma''_2} = 0.$$

Consequently, each element in the sum of equation (II.3) has a counterpart of opposite sign that cancels it. We conclude that $\delta^2_{\text{Morse}} = 0$. □

Define the Morse cohomology groups with coefficients in $\underline{A}$ by:

$$H^q_{\text{Morse}}(B, X; \underline{A}) := H^q(C^\bullet_{\text{Morse}}(B, X; \underline{A}), \delta_{\text{Morse}}).$$

Recall that the Stable/Unstable Manifold Theorem shows that the unstable manifolds of the negative gradient flow of a Morse function are open cells. In his work, Laudenbach [5] gives a condition that is sufficient to show that the $\overline{W}^u(x)$ are the cells of a CW-complex.

DEFINITION 4.12. The vector field X is said to be in standard form if, for every point $x \in \operatorname{Crit}(f)$, there exists a chart $(x^1, \ldots, x^n)$ around x for which X locally takes on the form $\sum_i x^i \frac{\partial}{\partial x^i}$.

When X is Morse-Smale and in standard form, Laudenbach was able to show that $\partial W^u(x) = \overline{W}^u(x) \setminus W^u(x)$ is a deformation retract of a neighbourhood of $\overline{W}^u(x)$. For all points x of index p, this defines an attaching map $S^{p-1} \to \bigcup_{\operatorname{ind} y < p} W^u(y)$ and thus a decomposition of B into a CW-complex with cells the unstable manifolds $W^u(x)$. Thus we have

$$\operatorname{Cell}^p(B; \underline{A}) = C^p_{\text{Morse}}(B, X; \underline{A})$$

as both are free abelian groups indexed over the critical points of index p.

In order to identify the two boundary operators, let $x, y \in \mathrm{Crit}(f)$, $\operatorname{ind} x = \operatorname{ind} y + 1$ and look at the number of connected components of $W^u(x)$ in a neighbourhood of $W^u(y)$. Theorem 4.9 tells us that the number is bounded from below by $\#\mathcal{M}(x,y)$, whereas Theorem 4.8 tells us that the number is bounded from above by $\#\mathcal{M}(x,y)$. As a consequence there are exactly $\#\mathcal{M}(x,y)$ many connected components and they are indexed by elements of $\mathcal{M}(x,y)$. For a fixed $\gamma \in \mathcal{M}(x,y)$, let $z := \gamma(t)$ for t sufficiently large such that z lies in the connected component indexed by γ. We have the isomomorphism of oriented vector spaces:

$$\frac{T_z W^u(x)}{\langle n_\gamma \dot{\gamma}(z)\rangle} = \frac{T_z W^u(x)}{T_z(W^u(x) \cap W^s(y))} \cong \frac{T_z B}{T_z W^s(y)} \overset{\text{par. transp.}}{\underset{\text{along } \gamma}{\cong}} \frac{T_y B}{T_y W^s(y)} \cong T_y W^u(y).$$

We deduce that $T_z W^u(x) \cong \langle n_\gamma \dot{\gamma}(z)\rangle \oplus T_y W^u(y)$. Since this is an oriented isomorphism, we can interpret it as follows: n_γ is equal to ± 1 depending on whether $W^u(y)$ inherits the orientation of the boundary of $W^u(x)$ or not. We define the incidence number

$$[W^u(y) : W^u(x)] = \sum_{\gamma \in \mathcal{M}(x,y)} n_\gamma.$$

Choosing the same orientations for the unstable manifolds as for the orientations of the cells in the description of the cellular boundary operator, we see by comparing (II.1) and (II.2) that :

$$\delta_{\mathrm{Cell}} = \delta_{\mathrm{Morse}}.$$

We gather the results as:

THEOREM 4.13 ([9, Thm. 3.2.18]). *Let X be a negative pseudo-gradient vector field on a compact manifold B. Suppose that X satisfies the Morse-Smale condition and that it is of standard form in a neighbourhood of the critical points. The unstable manifolds determine a CW-decomposition of B and for every local system $\underline{A}$, we have the isomorphism:*

$$H^p_{\mathrm{Morse}}(B, X; \underline{A}) \cong H^p_{\mathrm{Cell}}(B; \underline{A}) \qquad p \geq 0.$$

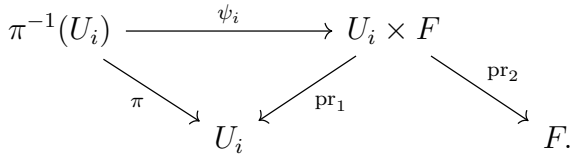
COROLLARY 4.14. *Let B,X and $\underline{A}$ be as above. We have an isomorphism that depends solely on the choice of orientations of the unstable manifolds*

$$H^p_{\mathrm{Morse}}(B, X; \underline{A}) \cong H^p(B; \underline{A}) \qquad p \geq 0.$$

5 Morse Functions on the Total Space of a Fibre Bundle

Let $f : B \to \mathbb{R}$ be a Morse function on the compact manifold B and let $\{p_1, \ldots, p_k\}$ be its critical points. The lift $\tilde{f} := f \circ \pi$ onto E is a Morse-Bott function as $\mathrm{Crit}(\tilde{f})$ is a disjoint union of finitely many connected submanifolds of E, namely the fibres over the critical points.

The idea is to perturb $\tilde{f}$ in a neighbourhood of these critical fibres such that we obtain a Morse function. Choose open neighbourhoods $p_i \in U_i$ such that $U_i \cap U_j = \emptyset$ for $i \neq j$ with trivialisations:



Fix a Morse function $\varphi : F \to \mathbb{R}$ and set $f_i := \varphi \circ \mathrm{pr}_2 \circ \psi_i : \pi^{-1}(U_i) \to \mathbb{R}$. Let $p_i \in V_i \Subset U_i$ be open subsets that are relatively compact, i.e. the closure of V_i is compact in U_i. Let $\rho_i : U_i \to \mathbb{R}$ be smooth truncation functions with compact support whose value is equal to 1 on V_i.

Proposition 5.1. *With the above notation, we have that the function*

$$f_\varepsilon := \tilde{f} + \varepsilon \sum_i \tilde{\rho}_i f_i$$

is a Morse function for all $\varepsilon > 0$ sufficiently small. Its critical points are (p_i, y) for $p_i \in \mathrm{Crit}(f)$, $y \in \mathrm{Crit}(f_i)$.

Proof. One can see that since ρ_i is zero outside of the open sets U_i and $\tilde{f}$ has no critical points outside of the $\pi^{-1}(U_i)$'s, all of the critical points of f_ε lie in $\bigcup_i \pi^{-1}(U_i)$. With the trivialisations ψ_i we can write:

$$\begin{aligned} f_\varepsilon(x,y) &= f(x) + \varepsilon \rho_i(x)\varphi(y) \\ Df_\varepsilon(x,y) &= (Df(x) + \varepsilon D\rho_i(x)\varphi(y), \varepsilon \rho_i(x) D\varphi(y)). \end{aligned}$$

As $Df \neq 0$ on $\overline{U}_i \setminus V_i$ since it contains no critical points and because φ is bounded, we can choose $\varepsilon > 0$ sufficiently small such that the first component of Df_ε is never zero on $\pi^{-1}(\overline{U}_i \setminus V_i)$. The truncation maps ρ_i are constantly equal to 1 on V_i, $Df_\varepsilon(x,y) = (Df(x), \varepsilon D\varphi(y))$ on $\pi^{-1}(V_i)$ and the critical points are (p_i, y), $y \in \mathrm{Crit}(\varphi)$. The non-degeneracy of the critical points is clear from the local expression. □

The rest of this chapter will be devoted to proving the following theorem:

Theorem 5.2 ([9, Thm. 3.3.1]). *Let $F \hookrightarrow E \xrightarrow{\pi} B$ be a fibre bundle, where B and F are compact manifolds without a boundary. Let $f : B \to \mathbb{R}$ be a Morse function. For $\varepsilon > 0$, there exists a Morse function $f_\varepsilon : E \to \mathbb{R}$ such that $||f_\varepsilon - f \circ \pi||_{C^1} \leq \varepsilon$ and a vector field X_ε that is a negative pseudo-gradient with respect to f_ε. Furthermore, X_ε satisfies that $\{X_\varepsilon = 0\} = \mathrm{Crit}(f_\varepsilon)$ and*

$$\pi(\mathrm{Crit}(f_\varepsilon)) = \mathrm{Crit}(f).$$

In addition:

a) There exists a spectral sequence $(E_r^{\bullet,\bullet})_{r \geq 1}$ that converges to $H^*_{\mathrm{Morse}}(E, X_\varepsilon; \mathbb{Z})$ and such that

$$\begin{aligned} E_1^{p,q} &= C^p_{\mathrm{Morse}}(B; \underline{H}^q_{\mathrm{Morse}}(F)), \quad d_1 = \delta_{\mathrm{Morse}}, \\ E_2^{p,q} &= H^p_{\mathrm{Morse}}(B; \underline{H}^q_{\mathrm{Morse}}(F)). \end{aligned}$$

The notation $\underline{H}_{\text{Morse}}$ denotes the local system of coefficients isomorphic to the Serre local system $\underline{H}$ that is defined in Lemma 5.7. This spectral sequence is exclusively constructed in terms of Morse cohomology.

b) The spectral sequence $(E_r^{\bullet,\bullet})_{r\geq 2}$ is isomorphic to the Serre spectral sequence $({}_{LS}E_r^{\bullet,\bullet})_{r\geq 2}$ by an isomorphism that only depends on a choice of orientations of the unstable manifolds of X_ε.

5.1 Adapted Metrics and Pseudo-Gradient Vector Fields

Let $V_\alpha := \ker(D\pi(\alpha) : T_\alpha E \to T_{\pi(\alpha)}B)$ denote the vertical distribution. By definition, a connection, sometimes called Ehresmann connection, on $\pi : E \to B$ is another distribution of rank equal to $\dim B$ that is transverse to the vertical distribution. We also sometimes call it a horizontal distribution. Note that the distribution V is canonical, but for the horizontal distribution there is a choice involved. Given a Riemannian metric ρ on E, one possible connection is given by $H_\rho := (V_\alpha^\perp)_{\alpha\in E}$.

The goal of the following is to, for $\varepsilon > 0$ sufficiently small as in Proposition 5.1, construct a negative pseudo-gradient vector field X_ε with respect to f_ε that is in standard form, verifies the Morse-Smale condition and satisfies

$$\pi_* X_\varepsilon = -\nabla f.$$

Define

$$X_\varepsilon := -\nabla \tilde{f} - \varepsilon \sum \tilde{\rho}_i \nabla f_i. \tag{II.4}$$

Proposition 5.3. *Let f, φ, f_i and ρ_i be as in Proposition 5.1. Let g be a metric on B and h be a metric on F. We also denote by h the metric induced on the vertical distribution of $U_i \times F$.*

a) The set $\mathcal{R}_{g,h}$ of metrics ρ on E such that $\rho_{|H_\rho} = \pi^* g$, $(\psi_i)_* H_\rho$ is the trivial horizontal distribution given by $\ker Dpr_2$ on $U_i \times F$ and $\rho_{|V} = \psi_i^* h$ on $\pi^{-1}(U)$ is non-empty;

b) There exists $\varepsilon_0 > 0$ such that for every metric $\rho \in \mathcal{R}_{g,h}$ the vector field X_ε defined by (II.4) is a negative pseudo-gradient vector field for f_ε and verifies $\pi_* X_\varepsilon = -\nabla f$ for all $0 < \varepsilon \leq \varepsilon_0$;

c) If $\nabla^g f$ and $\nabla^h \varphi$ are of standard form, then so is X_ε for all $\varepsilon > 0$.

Proof. a) By a partition of unity argument, construct a metric ρ_0 on E that agrees with $\psi_i^*(g \otimes h)$ on $\pi^{-1}(V_i)$. By construction, the horizontal distribution $H = H_{\rho_0}$ is then flat over V_i, i.e. $(\psi_i)_* H_\rho$ is the trivial horizontal distribution on the trivialisations. Define a metric ρ by:

$$\rho_{|V} = \rho_{0|V} \qquad \rho_{|H} = \pi^* g \qquad V \perp_\rho H.$$

Thus $\rho \in \mathcal{R}_{g,h}$.

b) Since $\tilde{f} = f \circ \pi$ is constant on the fibres and $f_i = \varphi \circ \mathrm{pr}_2 \circ \psi_i$ only depends on the fibre, we have:

$$D\tilde{f}_{|V} = 0 \text{ and } Df_{i|H} = 0.$$

By the definition of the gradient, we deduce:

$$\nabla^\rho \tilde{f} \in H \text{ and } \nabla^\rho f_i \in V.$$

Let Y be an arbitrary vector field and let Y_H be its horizontal lift. Then Y and Y_H are π-related. We claim that $\nabla^g f$ and $\nabla^\rho \tilde{f}$ are also π-related. Fix $b \in B$ and choose $e \in E_b$, then:

$$\begin{aligned}\rho_e(\nabla\tilde{f}(e), Y_H(e)) &= d\tilde{f}_{|e}(Y_H(e)) = Y_H(e)(f \circ \pi) = D\pi(e)[Y_H(e)](f) = \\ &= Y(\pi(e))(f) = Y(b)(f) = df_{|b}(Y(b)) = g_b(\nabla^g f(b), Y(b)).\end{aligned}$$

But since $\nabla\tilde{f} \in H$, we also have:

$$\begin{aligned}\rho_e(\nabla\tilde{f}(e), Y_H(e)) &= \rho_e(\nabla\tilde{f}(e)_H, Y_H(e)) = g_b(D\pi(e)[\nabla\tilde{f}(e)], D\pi(e)[Y_H(e)]) \\ &= g_b(D\pi(e)[\nabla\tilde{f}(e)], Y(b)).\end{aligned}$$

The fact that Y was chosen arbitrarily then proves the claim. Analogously, one shows that $\nabla\tilde{\rho}_i \in H$ and $\nabla\tilde{\rho}_i$ and $\nabla\rho$ are π-related.

As $\nabla f_i \in V$, we have $D\pi[X_\varepsilon] = -D\pi[\nabla\tilde{f}]$, and thus the push-forward $\pi_* X_\varepsilon$ is well-defined and equal to $-\nabla f$. Let us now evaluate $X_\varepsilon \cdot f_\varepsilon$. Since $X_\varepsilon = -\nabla f_\varepsilon$ above $(\cup V_i) \bigcup (\cup U_i)^c$, it is enough to work over $\overline{U}_i \setminus V_i$ where:

$$\begin{aligned}&(-\nabla\tilde{f} - \varepsilon \sum \tilde{\rho}_i \nabla f_i) \cdot (\tilde{f} + \varepsilon\tilde{\rho}_i f_i) \\ &= -\left(|\nabla\tilde{f}|_\rho + \varepsilon f_i g(\nabla\rho_i, \nabla f) + \varepsilon\tilde{\rho}_i\rho(\nabla\tilde{f}, \nabla f_i) + \varepsilon\tilde{\rho}_i\rho(\nabla\tilde{f}, \nabla f_i) + \varepsilon^2\tilde{\rho}_i f_i \rho(\nabla f_i, \nabla\tilde{\rho}_i) + \varepsilon^2\tilde{\rho}_i^2 |\nabla f_i|_\rho^2\right) \\ &\overset{V \perp_\rho H}{=} -\left(|\nabla f|_g + \varepsilon f_i g(\nabla\rho_i, \nabla f) + \varepsilon^2\tilde{\rho}_i^2 |\nabla f_i|_\rho^2\right) \\ &\leq -|\nabla f|_g^2 - \varepsilon f_i g(\nabla\rho_i, \nabla f) < 0.\end{aligned}$$

The last inequality holds for $\varepsilon > 0$ sufficiently small, as ∇f is never zero on the compact set $\overline{U}_i \setminus V_i$ as it contains no critical points of f.

c) The assertion follows from the structure of $\mathcal{R}_{g,h}$.

□

5.2 The Spectral Sequence for the TSW complex of a filtration

Fix f, ψ_i, ρ_i and φ as in the previous section together with metrics g and h on B and F such that $\nabla^g f$ and $\nabla^h \varphi$ are Morse-Smale vector fields that are in standard form. Fix a generic metric $\rho \in \mathcal{R}_{g,h}$ and $0 < \varepsilon \leq \varepsilon_0$. The vector field X_ε defined in (II.4) is then a negative pseudo-gradient vector field for f_ε, of Morse-Smale type and in standard form. Denote by $(C^\bullet, \partial)$ the TSW complex on $\mathbb{Z}$ associated to X:

$$C^k = \bigoplus_{\substack{\alpha \in \mathrm{Crit}(f_\varepsilon) \\ \mathrm{ind}\,\alpha = k}} \mathbb{Z}\langle\alpha\rangle.$$

Fix orientations for the unstable manifolds of f and φ. Define an orientation of a critical point $\alpha = (p_i, y)$ of f_ε to be the sum $\langle p_i \rangle \oplus \langle y \rangle$.

The total complex $C^\bullet$ naturally possess a descending filtration that respects the grading:

$$C_p = \bigoplus_{\substack{\alpha \in \mathrm{Crit}(f_\varepsilon) \\ \mathrm{ind}\,\pi(\alpha) \geq p}} \mathbb{Z}\langle \alpha \rangle.$$

PROPOSITION 5.4. *The differential ∂ preserves the filtration*

$$\partial(C_p^k) \subset C_p^{k+1}.$$

Proof. This follows from the fundamental property that $\pi_* X_\varepsilon = -\nabla f$. Thus, paths in the trajectory space of X_ε project onto trajectories of $-\nabla f$. If $\mathcal{M}(\alpha, \beta) \neq \emptyset$, $\alpha, \beta \in \mathrm{Crit}(f_\varepsilon)$, then $\mathcal{M}(\pi(\alpha), \pi(\beta)) \neq \emptyset$. By the Morse-Smale transversality condition on ∇f, this implies that $\mathrm{ind}\,\pi(\alpha) \geq \mathrm{ind}\,\pi(\beta)$ with equality if and only if $\pi\alpha = \pi\beta$. This means that both points and the trajectory connecting them lie in the same fibre. □

We are now going to define an exact couple:

$$A := \bigoplus_{p \geq 0} C_p$$
$$B := \bigoplus_{p \geq 0} C_p / C_{p+1} = \bigoplus_{p \geq 0} \bigoplus_{\substack{\alpha \in \mathrm{Crit}(f_\varepsilon) \\ \mathrm{ind}\,\pi(\alpha) = p}} \mathbb{Z}\langle \alpha \rangle.$$

The complex C_p/C_{p+1} can be identified with the total Morse complex of the union of all the fibres above the critical points of index p of the base space B. As free $\mathbb{Z}$-modules, this identification is immediate since we index over all critical points of $f_\varepsilon : E \to \mathbb{R}$ that, when projected onto B, are critical points of $f : B \to \mathbb{R}$ of index p, i.e. all critical points that lie in the critical fibres.

On the other hand, the induced differential $\tilde{\partial} : C_p/C_{p+1} \to C_p/C_{p+1}$ acts as

$$\tilde{\partial}[a] = \partial a + C_{p+1}, \qquad a \in C_p.$$

To identify the boundary operators, it suffices to check on the generators $\langle \alpha \rangle$, $\alpha \in \mathrm{Crit}(f_\varepsilon)$, $\mathrm{ind}\,\pi(\alpha) = p$. Suppose further that $\mathrm{ind}\,\alpha = k$. Denote by $\#\mathcal{M}(\alpha, \beta) = \sum_{\gamma \in \mathcal{M}(\alpha,\beta)} n_\gamma$. Then

$$\partial\langle \alpha \rangle = \sum_{\mathrm{ind}\,\beta = k+1} \#\mathcal{M}(\beta, \alpha)\langle \beta \rangle = \sum_{i \geq p} \sum_{\substack{\mathrm{ind}\,\beta = k+1 \\ \mathrm{ind}\,\pi(\beta) = i}} \#\mathcal{M}(\beta, \alpha)\langle \beta \rangle$$
$$= \sum_{\substack{\mathrm{ind}\,\beta = k+1 \\ \mathrm{ind}\,\pi(\beta) = \mathrm{ind}\,\pi(\alpha)}} \#\mathcal{M}(\beta, \alpha)\langle \beta \rangle + \sum_{i > p} \sum_{\substack{\mathrm{ind}\,\beta = k+1 \\ \mathrm{ind}\,\pi(\beta) = i}} \#\mathcal{M}(\beta, \alpha)\langle \beta \rangle,$$

where the last equality used that $\operatorname{ind}\pi(\beta) = p\pi(\alpha)$ implies that $\mathcal{M}(\beta,\alpha) \neq \emptyset$ if and only if $\pi(\alpha) = \pi(\beta)$. As seen in the proof of Proposition 5.4, this path then lies entirely in the fibre above $\pi(\alpha)$, and we get:

$$\begin{aligned}\partial\langle\alpha\rangle + C_{p+1} &= \sum_{\substack{\operatorname{ind}\beta=k+1\\ \operatorname{ind}\pi(\beta)=\operatorname{ind}\pi(\alpha)}} \#\mathcal{M}(\beta,\alpha)\langle\beta\rangle + C_{p+1}\\ &= \sum_{\substack{\operatorname{ind}_{\text{fibre}}\beta=\operatorname{ind}_{\text{fibre}}\alpha+1\\ \beta\in F_{\pi(\alpha)}}} \#\mathcal{M}_{\text{fibre}}(\beta,\alpha)\langle\beta\rangle + C_{p+1}\\ &= \partial_{\text{fibre}}\langle\alpha\rangle + C_{p+1},\end{aligned}$$

where we denote the Morse boundary operator of the fibre $F_{\pi(\alpha)}$ by ∂_{fibre}. Consequently:

$$E_1 = B_1 \cong \bigoplus_{p\geq 0} H_{\text{Morse}}\Big(\bigsqcup_{\substack{p_i\in B\\ \operatorname{ind}p_i=p}} F_{p_i}\Big) = \bigoplus_{p\geq 0}\bigoplus_{\substack{p_i\in B\\ \operatorname{ind}p_i=p}} H_{\text{Morse}}(F_{p_i}). \tag{II.5}$$

We are now going to give an explicit description of the differential $d_1 = j_1 \circ k_1 : E_1 \to E_1$. Define $C'_p = C_p/C_{p+1} = \bigoplus_{\operatorname{ind}\pi(\alpha)=p}\mathbb{Z}\langle\alpha\rangle$. Then

$$C_p = C'_p \bigoplus C_{p+1}.$$

There is an induced boundary operator:

$$\begin{aligned}&\partial'' : C'_p \to C_{p+1}\\ &\partial''(\alpha) = \sum_{\substack{\operatorname{ind}\beta=\operatorname{ind}\alpha+1\\ \operatorname{ind}\pi(\beta)\geq\operatorname{ind}\pi(\alpha)+1}} \#\mathcal{M}(\beta,\alpha)\langle\beta\rangle \qquad \text{for } \alpha\in\operatorname{Crit}(f_\varepsilon).\end{aligned}$$

The induced boundary operator on the filtration then decomposes as:

$$\begin{aligned}&\partial_p : C_p = C'_p \oplus C_{p+1} \to C_p\\ &\begin{cases} C'_p & \xrightarrow{\partial_{\text{fibre}}\oplus\partial''} C'_p \oplus C_{p+1}\\ C_{p+1} & \xrightarrow{0\oplus\partial_{p+1}} C'_p \oplus C_{p+1}\end{cases}\end{aligned} \tag{II.6}$$

A cohomology class in $H(C_p)$ is represented by an element $\alpha' \oplus \alpha'' \in C'_p \oplus C_{p+1}$. The inclusion $C_{p+1} \hookrightarrow C_p$ is written as:

$$\underbrace{\alpha'}_{C'_{p+1}} \oplus \underbrace{\alpha''}_{C_{p+2}} \mapsto \underbrace{0}_{C'_p} \oplus \underbrace{(\alpha' \oplus \alpha'')}_{C_{p+1}}$$

i_1 is then defined by letting the cohomology functor act on this map.

On the other hand, since we were able to identify $\partial\langle\alpha\rangle$ with $\partial_{\text{fibre}}\langle\alpha\rangle$, it is possible to define j_1 by taking the cohomology of $\alpha' \in C'_p$ with respect to ∂_{fibre}. Furthermore,

this entails a well-defined map between $H(A)$ and $H(B)$, since $\partial_p(\alpha' \oplus \alpha'') = 0$ implies that $\partial_{\text{fibre}}(\alpha') = 0$ by (II.6).

$$\begin{aligned} j_1 : H(A) = \bigoplus_p H(C_p) &\to H(B) = \bigoplus_p H(C_p/C_{p+1}) \\ [\alpha' \oplus \alpha'']_{\partial_p} &\mapsto [\alpha']_{\partial_{\text{fibre}}}. \end{aligned}$$

In order to define k_1 it is enough to look at the k-th degree of the exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_{p+1}^{k+1} & \xrightarrow{\ i\ } & C_p^{k+1} & \longrightarrow & (C_p/C_{p+1})^{k+1} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow{\scriptstyle \partial_p} & & \uparrow & & \\ 0 & \longrightarrow & C_{p+1}^{k} & \longrightarrow & C_p^{k} & \longrightarrow & (C_p/C_{p+1})^{k} & \longrightarrow & 0. \end{array}$$

Let $\alpha = \sum_i \alpha_i \in (C_p/C_{p+1})^k$ where each α_i is a linear combination with integer coefficients of critical points in a fibre F_{p_i} with $\operatorname{ind} p_i = p$. Since

$$\partial_p \alpha = 0 \Leftrightarrow \partial_{\text{fibre}} \alpha = 0 \Leftrightarrow \forall i, \partial_{\text{fibre}} \alpha_i = 0$$

we can suppose by linearity that $\alpha = \alpha_i$ is equal to a single critical point.

$$\begin{aligned} \partial_p(\alpha_i, 0) = (0, \partial'' \alpha_i) &= (0, \sum_{\substack{\operatorname{ind}\beta = k+1 \\ \operatorname{ind}\pi(\beta) \geq p+1}} \#\mathcal{M}(\beta, \alpha_i)\langle\beta\rangle) \\ &= i\Big(\sum_{\substack{\operatorname{ind}\beta = k+1 \\ \operatorname{ind}\pi(\beta) = p+1}} \#\mathcal{M}(\beta, \alpha_i)\langle\beta\rangle, \sum_{\substack{\operatorname{ind}\beta = k+1 \\ \operatorname{ind}\pi(\beta) \geq p+2}} \#\mathcal{M}(\beta, \alpha_i)\langle\beta\rangle \Big). \end{aligned}$$

The morphism k_1 is defined by taking the cohomology class of the last pair. After composition with j_1, we obtain

$$\begin{aligned} j_1 \circ k_1 = d_1 : E_1 &\to E_1, \\ \underbrace{[\alpha]}_{\substack{\operatorname{ind}\alpha = k \\ \operatorname{ind}\pi(\alpha) = p}} &\mapsto \Big[\sum_{\substack{\operatorname{ind}\beta = k+1 \\ \operatorname{ind}\pi(\beta) = p+1}} \#\mathcal{M}(\beta, \alpha)\langle\beta\rangle \Big]. \end{aligned}$$

In the above formula, a representative α of the class $[\alpha]$ is a linear combination with integer coefficients of critical points that lie in the same fibre above a critical point of index p. For a fixed critical point β with $\operatorname{ind}\beta = k+1$, the integer $\#\mathcal{M}(\beta, \alpha)$ represents the extension by linearity in the second argument of the quantity $\#\mathcal{M}(\beta, \alpha_i)$ defined for the critical points α_i of index k.

In order to make the expression of d_1 a little more appealing, define the map:

$$\begin{aligned} \Phi_\gamma^{\bar{x},x} : C^q_{\text{Morse}}(F_x) &\to C^q_{\text{Morse}}(F_{\bar{x}}) \\ \langle y \rangle &\mapsto \sum_{\operatorname{ind} z = q} \#\mathcal{M}_\gamma(\bar{x} \otimes z, x \otimes y)\langle z \rangle \end{aligned}$$

where $x, \bar{x} \in B$, $\operatorname{ind} x = p$, $\operatorname{ind}\bar{x} = p+1$, $C^q_{\text{Morse}}(F_x)$ and $C^q_{\text{Morse}}(F_{\bar{x}})$ are the Morse complexes in the fibres F_x and $F_{\bar{x}}$, $\gamma \in \mathcal{M}(\bar{x}, x)$ and

$$\mathcal{M}_\gamma = \text{trajectories in } E \text{ that project onto } \gamma \text{ in } B.$$

The notation $x \otimes y$ relates to the point y of the fibre F_x as a point in E, and is a tool for keeping track of basepoints.

LEMMA 5.5. *The map $\Phi_\gamma^{\bar{x},x}$ is a chain map, i.e.:*

$$\Phi_\gamma^{\bar{x},x} \circ \partial_x + \partial_{\bar{x}} \circ \Phi_\gamma^{\bar{x},x} = 0$$

where $\partial_x, \partial_{\bar{x}}$ represent the Morse differentials in the fibres F_x and $F_{\bar{x}}$.

Proof. Let $y' \in F_x$, $\operatorname{ind}_{\text{fibre}}(y') = q - 1$. We have:

$$\begin{aligned}
\Phi_\gamma^{\bar{x},x} \circ \partial_x \langle y' \rangle &= \Phi_\gamma^{\bar{x},x}\Big(\sum_{\operatorname{ind} y = q} \#\mathcal{M}_{F_x}(y, y') \langle y' \rangle \Big) \\
&= \sum_{\operatorname{ind} y = q} \sum_{\operatorname{ind} z = q} \#\mathcal{M}_{F_x}(y, y') \cdot \#\mathcal{M}_\gamma(\bar{x} \otimes z, x \otimes y) \langle z \rangle \\
&= \sum_{\operatorname{ind} z = q} \left(\sum_{\operatorname{ind} y = q} \#\mathcal{M}_{F_x}(y, y') \cdot \#\mathcal{M}_\gamma(\bar{x} \otimes z, x \otimes y) \right) \langle z \rangle
\end{aligned}$$

and in a similar manner:

$$\partial_{\bar{x}} \circ \Phi_\gamma^{\bar{x},x} \langle y' \rangle = \sum_{\operatorname{ind} z = q} \left(\sum_{\operatorname{ind} z' = q-1} \#\mathcal{M}_{F_{\bar{x}}}(z, z') \cdot \#\mathcal{M}_\gamma(\bar{x} \otimes z', x \otimes y') \right) \langle z \rangle.$$

It is thus enough to show that fox fixed $z \in \operatorname{Crit}_q(F_{\bar{x}})$ we have:

$$c_\gamma := \sum_{\operatorname{ind} y = q} \#\mathcal{M}_{F_x}(y, y') \cdot \#\mathcal{M}_\gamma(\bar{x} \otimes z, x \otimes y) + \sum_{\operatorname{ind} z' = q-1} \#\mathcal{M}_{F_{\bar{x}}}(z, z') \cdot \#\mathcal{M}_\gamma(\bar{x} \otimes z', x \otimes y') = 0.$$

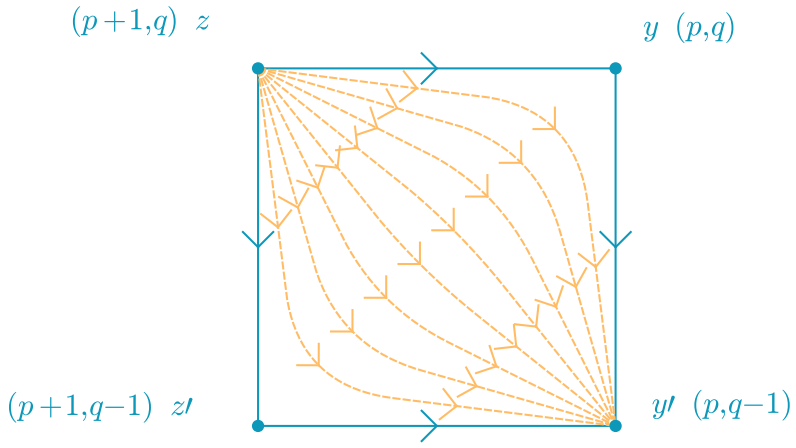
Note that a one-parameter family of trajectories linking $\bar{x} \otimes z$ to $x \otimes y'$ projects onto a one-parameter family linking $\bar{x}$ to x. But $\mathcal{M}(\bar{x}, x)$ is a finite set since $\operatorname{ind} \bar{x} = \operatorname{ind} x + 1$, which ensures by continuity that every one-parameter family in $\mathcal{M}(\bar{x} \otimes z, x \otimes y')$ projects onto a unique element of $\mathcal{M}(\bar{x}, x)$ This implies that when we sum c_γ over all $\gamma \in \mathcal{M}(\bar{x}, x)$, we never count trajectories in the total space twice. Thus we obtain:

$$\sum_{\gamma \in \mathcal{M}(\bar{x}, x)} c_\gamma = \sum_{\operatorname{ind} \alpha = p+q} \#\mathcal{M}(\bar{x} \otimes z, \alpha) \cdot \#\mathcal{M}(\alpha, x \otimes y').$$

But this expression counts broken trajectories that for $\alpha \in \operatorname{Crit}_{p+q}(f_\varepsilon)$ flow from $\bar{x} \otimes z$ over α to $x \otimes y'$. By Proposition 4.9, broken trajectories form the boundary of $\mathcal{M}(\bar{x} \otimes z, x \otimes y')$ since $\operatorname{ind}(\bar{x} \otimes z) = \operatorname{ind}(x \otimes y') + 2$. Consequently

$$\sum_{\gamma \in \mathcal{M}(x, y)} c_\gamma = \sum_{\sigma \in \partial \mathcal{M}(\bar{x} \otimes z, x \otimes y')} \operatorname{sgn}(\sigma) = 0$$

by Proposition 4.10, as for each boundary is composed of two broken trajectories of opposite orientation.



By the previous remark, these pairs of broken trajectories are then located in the same sum c_γ as we can find a one-parameter family of trajectories in the total space that links the two broken trajectories. Thus they must both project onto γ. □

Denote also by a slight abuse of notation $\Phi_\gamma^{\bar{x},x}$ for the map induced in cohomology. The differential d_1 takes on the following form:

$$d_1 : x \otimes [u] \mapsto \sum_{\operatorname{ind} \bar{x}=p+1} \bar{x} \otimes \sum_{\gamma \in \mathcal{M}(\bar{x},x)} \Phi_\gamma^{\bar{x},x}([u]) \tag{II.7}$$

with $\operatorname{ind} x = p$, $u \in C^q(F_x)$, $\partial_{\text{fibre}} u = 0$.

LEMMA 5.6. *The map $\Phi_\gamma^{\bar{x},x} : C^q(F_x) \to C^q(F_{\bar{x}})$ induces an isomorphism on cohomology.*

Proof (Sketch). We are going to give a definition of $\Phi_\gamma^{\bar{x},x}$ as a chain map induced by a certain "deformation" of f_x into $f_{\bar{x}}$. As $\operatorname{im}\gamma$ is contractible, the pullback bundle over γ is trivial:

$$\begin{aligned} \psi : E_{|\operatorname{im}\gamma} &\xrightarrow{\cong} \mathbb{R} \times F \\ F_{\gamma(t)} &\mapsto \{-t\} \times F \end{aligned}$$

that coincides with ψ_x on $\operatorname{im}\gamma \cap U_x$ and $\psi_{\bar{x}}$ on $\operatorname{im}\gamma \cap U_{\bar{x}}$ respectively under the projection $\operatorname{im}\gamma \to \mathbb{R}$, $\gamma(t) \mapsto -t$. In this trivialisation, we can interpret $H = f_\varepsilon \circ \psi^{-1}$ as a deformation from $\varepsilon\varphi = f_\varepsilon \circ \psi_x^{-1}$ to $\varepsilon\varphi = f_\varepsilon \circ \psi_{\bar{x}}^{-1}$:

$$\begin{aligned} H &= (H_t)_{t\in\mathbb{R}} : F \to \mathbb{R} \\ H &= \varepsilon\varphi \quad \text{in the limit } \pm\infty. \end{aligned}$$

By considering the metric g_t of $F_{\gamma(-t)}$ induced by ψ on $\{t\} \times F$, the vector field X_ε can be written on $\mathbb{R} \times F$ as:

$$X_\varepsilon = -\nabla_{g_t} H_t + \frac{\partial}{\partial t}$$

and constitutes a deformation of $\varepsilon\nabla\varphi$. If Φ_s denotes the flow of $-\nabla_{g_t} H_t + \frac{\partial}{\partial t}$, define for $y \in \operatorname{Crit}(\varphi)$ the unstable manifold:

$$\mathcal{U}_y := \{(t,z) \in \mathbb{R} \times F \mid \lim_{s\to-\infty} \Phi_s(t,z) = (-\infty, y)\}$$

and the stable manifold:

$$\mathcal{S}_y := \{(t,z) \in \mathbb{R} \times F \mid \lim_{s\to+\infty} \Phi_s(t,z) = (+\infty, y)\}$$

of dimension $\operatorname{ind} y + 1$ and $\dim F - \operatorname{ind} y + 1$ respectively. The intersections $\mathcal{M}(y,z) = \mathcal{U}_y \cap \mathcal{S}_z$ are transverse for $y \neq z$ and of dimension $\in y = \operatorname{ind} z + 1$. The group $\mathbb{R}$ acts by translation on $\widetilde{\mathcal{M}}(y,z)$ and $\mathcal{M}(y,z) := \widetilde{\mathcal{M}}(y,z)/\mathbb{R}$ is finite for $\operatorname{ind} y = \operatorname{ind} z$. In [12, §2.3] Austin and Braam prove that

$$\begin{aligned} \chi : C^q(F) &\to C^q(F) \\ \langle y \rangle &\mapsto \sum_{\operatorname{ind} z = q} \#\mathcal{M}(y,z) \langle z \rangle \end{aligned}$$

is a chain map that on the level of cohomology induces a morphism independent of the deformation with fixed ends. In our case, the trivial deformation $H'_t := \varepsilon\varphi$, $X'_\varepsilon := -\varepsilon\nabla\varphi + \frac{\partial}{\partial t}$ determines $\chi' = \mathrm{Id}$ on the level of chain complexes, and thus $H^\bullet(\chi') = \mathrm{Id}$ in cohomomology. Thus $H^\bullet(\chi) = \mathrm{Id}$ in cohomology.

On the other hand, from its definition one can see that χ is conjugated to $\Phi_\gamma^{\bar{x},x}$ by the trivialisations ψ_x and $\psi_{\bar{x}}$. This implies that $\Phi_\gamma^{\bar{x},x}$ induces an isomorphism in cohomology. □

The map $\Phi_\gamma^{\bar{x},x}$ presents strong similarities to the definition of a local system, and indeed we have the following result. The transport in the fibres along a path γ coincides, module identification of Morse and singular cohomology, with the transport along γ of a fibre by the Serre local system of 2.34. More precisely:

LEMMA 5.7. *Let γ be a path in B from $\bar{x}$ to x. The following diagram commutes:*

$$\begin{array}{ccc} H^q_{\text{Morse}}(F_x) & \xrightarrow{\Phi_\gamma^{\bar{x},x}} & H^q_{\text{Morse}}(F_{\bar{x}}) \\ \downarrow{\scriptstyle A_x} & & \downarrow{\scriptstyle A_{\bar{x}}} \\ H^q(F_x) & \xrightarrow[\tau_\gamma^{\bar{x},x}]{} & H^q(F_{\bar{x}}) \end{array}$$

where $\tau_\gamma^{\bar{x},x}$ is the isomorphism of the Serre local system of Construction 2.34 and $A_x, A_{\bar{x}}$ are the canonical isomorphisms of Corollary 4.14 that depends on the orientations of the unstable manifolds of F_x and $F_{\bar{x}}$ induced by the trivialisations ψ_x and $\psi_{\bar{x}}$.

Proof. Two commutative diagrams yield the result:

$$\begin{array}{ccccc} H^q_{\text{Morse}}(F_x) & & \xrightarrow{\Phi_\gamma^{\bar{x},x}} & & H^q_{\text{Morse}}(F_{\bar{x}}) \\ & \searrow & & \swarrow & \\ \downarrow{\scriptstyle A_x} & & H^q_{\text{Morse}} & & \downarrow{\scriptstyle A_{\bar{x}}} \\ & & \downarrow{\scriptstyle A} & & \\ H^q(F_x) & \xrightarrow[(\psi_x^*)^{-1}]{} & H^q(F) & \xleftarrow[(\psi_{\bar{x}}^*)^{-1}]{} & H^q(F_{\bar{x}}) \end{array}$$

with the arrows pointing to the centre being induces by a trivialisation ψ of $E_{|\operatorname{im}\gamma}$ that coincides with ψ_x and $\psi_{\bar{x}}$ in a neighbourhood of x and $\bar{x}$ respectively, like in the proof of the above lemma. The upper triangle is also commutative by Lemma 5.6.

To finish the proof, we claim that the isomorphism $\psi_{\bar{x}}^* \circ (\psi_x^*)^{-1}$ of the lower line is equal to $\tau_\gamma^{\bar{x},x}$. Indeed, by letting U be an open ball containing $\operatorname{im}\gamma$ over which one extends the trivialisation ψ, we have the commutative diagram:

$$\begin{array}{ccccc} H^q(F_x) & \xleftarrow{i_x^*} & H^q(E_{|U}) & \xrightarrow{i_{\bar{x}}^*} & H^q(F_{\bar{x}}) \\ \uparrow & & \uparrow{\scriptstyle \Phi^*} & & \uparrow \\ H^q(\{x\} \times F) & \longleftarrow & H^q(U \times F) & \longrightarrow & H^q(\{\bar{x}\} \times F) \\ & \nwarrow & \uparrow{\scriptstyle \pi^*} & \nearrow & \\ & & H^q(F) & & \end{array}$$

where $i_x : F_x \hookrightarrow E$, $i_{\bar{x}} : F_{\bar{x}} \hookrightarrow E$ are the inclusions of the fibres. By Construction 2.34 $\tau_\gamma^{\bar{x},x} = i_{\bar{x}}^* \circ (i_x^*)^{-1}$ and traversing the diagram counter-clockwise yields $\psi_{\bar{x}}^* \circ \psi_x^{*-1}$. □

One last ingredient is needed to start the proof of Theorem 5.2.

Proposition 5.8 ([9, Thm. 3.3.9]).

a) There exists a unique local system on B where the fibres at the critical points of f are $H^q_{\mathrm{Morse}}(F_x)$, $x \in \mathrm{Crit}(f)$ and where the transport of the fibres along paths in B is given by $\Phi_\gamma^{\bar{x},x}$. We denote it by $\underline{H}^q_{\mathrm{Morse}}(F)$.

b) The system of isomorphisms $(A_x : H^q_{\mathrm{Morse}}(F_x) \xrightarrow{\cong} H^q(F_x))_{x\in\mathrm{Crit}(f)}$ extends uniquely to an isomorphism of the local systems $\underline{H}^q_{\mathrm{Morse}}(F)$ and the Serre local system $\underline{H}^q(F)$.

Proof of Theorem 5.2. By now all the heavy-lifting is done and the proof is merely a matter of gathering the results. The requirements on f_ε and X_ε are clear by construction. On the topic of a), the calculation of E_1 was done in equation (II.5) and the identification $d_1 = \delta_{\mathrm{Cell}}$ is a direct consequence of equations (II.2) and (II.7). Proposition 5.8 ensures that $E_2^{p,q} \cong H^p(B; \underline{H}^q(F))$.

Let us now show b). The identification of Morse and cellular cohomology and of the local coefficient systems $\underline{H}^q_{\mathrm{Morse}}(F)$ and $\underline{H}^q(F)$ implies an isomorphism of the differential bicomplexes

$$(E_1^{\bullet,\bullet}, d_1) \cong ({}_{LS}E_1^{\bullet,\bullet}, {}_{LS}d_1).$$

We prove that both spectral sequences are isomorphic by showing it on the level of exact couples of rank $r = 1$. We have

$$\begin{aligned} {}_{LS}A_1 &= \bigoplus_{p\geq 0} H(E, E^{p-1}), \\ A_1 &= \bigoplus_{p\geq 0} H(C_p) \end{aligned}$$

and the isomorphism $H(C_p) \cong H(E, E^{p-1})$ results from the fact that the two members are isomorphic to the relative cellular homology $H_{\mathrm{Cell}}(E, E_{p-1})$ defined in Remark 4.5. The morphisms i_1, j_1 and k_1 as well as the terms B_1 were all made explicit, and are isomorphic to their counterparts. We conclude that both spectral sequences are isomorphic.

□

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