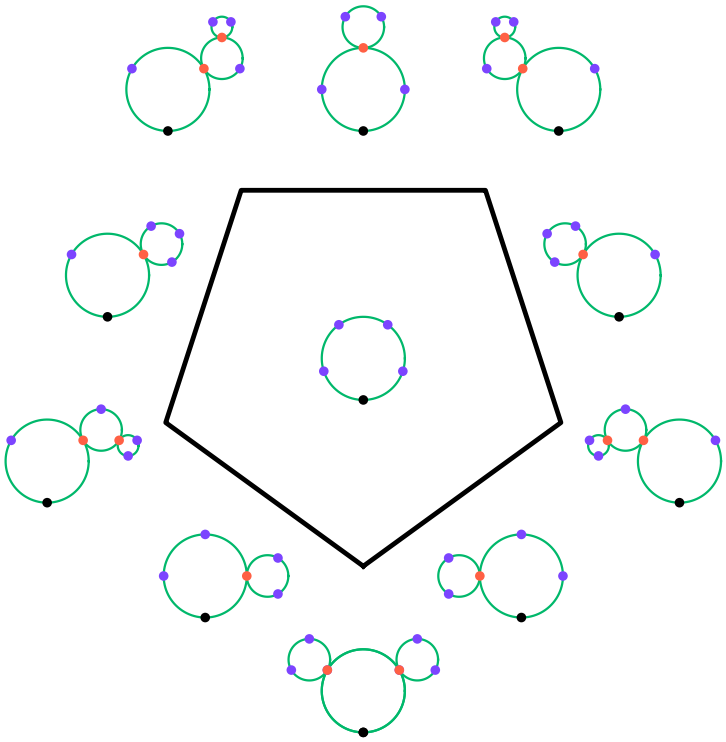
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Master Thesis

Viterbo's Theorem and the Wrapped Fukaya Category

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Introduction

Symplectic (co-)homology of the cotangent bundle is an invariant associated to the cotangent bundle of a closed manifold M that emerges from the study of a selected family of Hamiltonians whose 'growth rate' at infinity can be controlled. In many cases, this means that outside of a compact set, the Hamiltonian only depends on the radial distance r to the zero section inside of the bundle.

We will adopt the point of view presented in [Abo09] and assume that outside of the unit disc bundle $D^*(T^*M)$, the Hamiltonian depends linearly on r, i.e. that there exists a positive real number b, called the slope, such that $H^b \equiv b \cdot r$. Under certain favourable assumptions on the almost complex structure, one can then show that all the periodic orbits of the Hamiltonian vector field of the linear Hamiltonian H^b are entirely contained in the disc bundle. Hence it is possible to use standard Hamiltonian Floer theoretic results to assign a cohomology group to H^b for each slope b.

As will become clear later on, whenever $b' \leq b$ there exists a continuation map from the Floer cohomology group associated to $H^{b'}$ to the one associated to H^b. Symplectic cohomology, as defined by Abouzaid, is then the filtered colimit over these cohomology groups and it is denoted by $SH^*(T^*M; \mathbb{Z})$.

Cieliebak, Floer and Hofer [HF94; HFC95] were the first to introduce such groups in their study of Liouville domains through pseudo-holomorphic curves. In [HF94], they were able to show that there are families of Hamiltonians whose Floer cohomology groups are closely linked to the dynamics of the Reeb flow on the contact manifold at infinity. Inspired by the theory of generating functions, Viterbo was able to present such a class of Hamiltonians in [Vit99; Vit98]. For general Liouville domains, a definition of symplectic cohomology is given in [Sei06].

We will see later on that on the cotangent bundle, the Hamiltonian flow for the kinetic Hamiltonian $H(q,p) = |p|^2/2$ can be identified with the geodesic flow. As closed geodesics generate the homology of the free loop space $\mathcal{L}M$, a link between symplectic (co-)homology and $H_*(\mathcal{L}M)$ was conjectured.

Such a connection was indeed first presented by C. Viterbo at the ICM in Zurich [Vit94], and later on shown in [Vit99]. It states that the symplectic cohomology of the cotangent bundle is isomorphic to the homology of the free loop space with its grading reversed. Two analytically more involved, but overall more direct approaches were subsequently given by [AS06] and [SW06]. We will however follow

another variation given by M. Abouzaid in [Abo14]. A survey article comparing the different approaches is given by Weber [Web05].

As became apparent later on, the result as initially stated only holds true if the base is assumed to be Spin. This was noticed by T. Kragh [Kra07], who was able to show that the result still holds after introducing a local system of rank 1 on the free loop space that is trivial if and only if the second Stiefel-Whitney class vanishes.

The group isomorphism of Viterbo's theorem can be extended to a ring isomorphism. This is for instance done in [AS12] and [Abo14], where roughly speaking the ring structure on $H_*(\mathcal{L}M)$ is induced by the concatenation of loops. The first chapter of this thesis is dedicated to showing the following result:

Theorem 1 (Viterbo)**.** *Let M be a closed manifold. There exists a local coefficients system η on $\mathcal{L}M$ such that*

$$SH^*(T^*M;\mathbb{Z}) \cong H_*(\mathcal{L}M;\eta)$$

as rings.

In order to accomplish proving this result we will proceed as follows: Section 1 serves as a reminder of the most important concepts in symplectic geometry that we will use. In Sections 2 and 3, we define symplectic cohomology and the ring structure it exhibits in full detail. Subsequently in Section 4, we construct a finite approximation of the free loop space that functions as a finite model of it. Sections 5 and 6 then contain the construction of a map $\mathcal{F} : SH^*(T^*M;\mathbb{Z}) \to H_*(\mathcal{L}M)$ that will provide the isomorphism for Theorem 1.

While working on homological mirror symmetry, it became more and more apparent that Theorem 1 presented one major flaw: the resolution provided by passing to homology was too low; a lot of information is lost and it became clear that a similar result at the chain level would be desirable. Such a result was provided by M. Abouzaid in [Abo09], and Chapter 2 will be dedicated to showing the following spiritual descendent of Viterbo's theorem at the chain level:

Theorem 2 (Abouzaid)**.** *If M is a closed manifold, there exists an A_∞-equivalence*

$$CW_b^*(T^*M) \to C_{-*}(\Omega_q M)$$

between the homology of the space of loops on M based at q and the Floer cohomology of the cotangent fibre at q taken as an object of the wrapped Fukaya category of T^*M with background class $b \in H^*(T^*M;\mathbb{Z}_2)$ given by the pullback of $w_2(M) \in H^*(M;\mathbb{Z}_2)$.

The outline of the second chapter is such that we first define and provide intuition for A_∞-structures in Section 7. We follow this up by defining the A_∞-categories on both sides of the isomorphism of Theorem 2 in Section 8. The final sections then show that there exists an A_∞-functor between $CW_b^*(T^*M)$ and $C_{-*}(\Omega_q M)$, that is an A_∞-equivalence.

As a disclaimer, this thesis functions as an exposition on the two main references by M. Abouzaid. We will follow [Abo14] for Chapter 1 and [Abo09] for Chapter 2, and supplement the exposition presented therein with explanations wherever we deemed them appropriate.

Acknowledgement

At this point, I would like to express my sincerest gratitude towards my supervisor Prof. Will Merry. His (virtual) office door was always open to me and I would like to particularly thank him for introducing me to the wonderful world of Floer theories and, especially, the Fukaya category. Lastly, I am thankful for the many lectures of his that I attended; they were the source of a great deal of inspiration.

To my grandmother: Thérèse, you will be dearly missed.

Chapter I

Viterbo's Theorem — the Ring Version

1 A Primer on Symplectic Geometry

In this introductory section we first present how classical mechanics — in particular Hamiltonian mechanics — lies at the foundation of symplectic geometry. We then identify certain invariants and structures that can be generalised to manifolds, all the while recalling some basic symplectic geometry.

1.1 Motivation

The Hamiltonian formulation of classical mechanics describes a physical system in terms of generalised coordinates $(q_i)_{i\in\{1,\dots n\}}$ and conjugate momenta $(p_i)_{i\in\{1,\dots n\}}$. It turns out that the equations of motion of classical mechanics are equivalent to the following system of $2n$ first-order differential equations:

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \qquad \text{and} \qquad \dot{p}_k = -\frac{\partial H}{\partial q_k}. \tag{1.1.1}$$

for $H = T + V$, the sum of kinetic and potential energy. One often calls the collections of the q_i's and p_i's the *phase space*.

As it is often the case in physics, the Hamiltonian formalism can also be derived from the minimisation of some mysterious *action functional*. Let $\vec{z} : [0,1] \to \mathbb{R}^{2n}$ be a curve in phase space. Define the *action integral*:

$$\Phi_H(\vec{z}) := \int_0^1 (\langle \vec{p}, \dot{\vec{q}} \rangle - H(t, \vec{q}, \vec{p}))dt, \tag{1.1.2}$$

where $\vec{z}(t) = (\vec{q}, \vec{p})$ and $\langle \cdot, \cdot \rangle$ is the standard inner product. This is the integral of the so-called *action form*:

$$\lambda_H = \sum_{i=1}^{n} p_i dq_i - Hdt. \tag{1.1.3}$$

We have the following result:

PROPOSITION 1.1.1 ([**McSal**]). *A curve $\vec{z} : [0,1] \to \mathbb{R}^{2n}$ is a critical point of Φ_H (with respect to variations with fixed end points) if and only if it satisfies Hamilton's equations (1.1.1).* $\square_{\text{ref.}}$

Taking a closer look at the Hamiltonian system (1.1.1), it turns out that in the coordinates $z = (q_1, \ldots, q_n, p_1, \ldots, p_n) \in \mathbb{R}^{2n}$ the above system can be written in the form

$$J_0 \dot{z} = \nabla H(z),$$

where ∇H denotes the gradient of H and J_0 denotes the $2n \times 2n$ matrix

$$\begin{pmatrix} 0 & -\mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}.$$

Note that this is a rotation by 90° in phase space and that $J^2 = -\mathrm{id}$. This last fact presents close similarities to complex numbers[1] and it will be this structure together with (1.1.3) that we will generalise to manifolds. For a more detailed exposition on the origins of symplectic geometry (and a picture of the symplectic camel [**McSal**]) we refer to [**McSal**].

1.2 Symplectic Manifolds

When working with manifolds it is typically useful to first describe the desired structure locally, and afterwards patch the structures together in a consistent fashion. This is why we will first turn our attention to vector spaces.

DEFINITION 1.2.1. *A **symplectic vector space** is a vector space V over a field $\mathbb{K}$ (in our cases $\mathbb{R}$ or $\mathbb{C}$) equipped with a non-degenerate, alternating and bilinear form.*

The standard example on $\mathbb{R}^n \times \mathbb{R}^n$ is

$$\omega((p,q),(p',q')) = pq' - q'p = -(p,q) \cdot J_0 \cdot \begin{pmatrix} p' \\ q' \end{pmatrix}.$$

The next result shows that the dimension is the only invariant of the isomorphism type of symplectic vector spaces and that their dimension is even.

PROPOSITION 1.2.2 ([AD14, Prop. 5.1.1.]). *Let ω be a non-degenerate alternating bilinear form on the finite-dimensional vector space V. There exists a basis $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ such that*

$$\omega(e_i, f_j) = \delta_{ij} \text{ and } \omega(e_i, e_j) = \omega(f_i, f_j) = 0.$$

Proof. Since ω is non-degenerate, it is non-zero and we can find two vectors e_1 and f_1 that are linearly independent such that $\omega(e_1, f_1) = 1$. The form ω then restricts to a non-degenerate form on the orthogonal complement to $\mathrm{span}(e_1, f_1)$ and we conclude by induction on the dimension, after having noted that an alternating bilinear form on a space of dimension 1 is zero. $\square$

[1] It is in fact this similarity that motivated H. Weyl to introduce the word symplectic into the jargon of mathematics by substituting the Latin root in complex by the corresponding Greek root to label the symplectic group [Can01, p.1]

In order to generalise this idea to manifolds, we want to define a symplectic vector space structure on each tangent space. As we are working with manifolds, we in addition want this structure to vary smoothly. We thus endow our manifold with a non-degenerate differential 2-form ω.

The non-degeneracy condition already has a strong implication on the nature of our manifold M. Note that

$$\omega_x^{\wedge n} = \wedge^n \omega_x \neq 0 \in \bigwedge^{2n} T_x^* M \quad \forall x \in M.$$

Thus $\omega_x^{\wedge n}$ is a volume form on M, implying that M is orientable.

Another property that we want to satisfy is that the differential calculus that ω allows us to do on M is locally the same as on $\mathbb{R}^n \times \mathbb{R}^n$. Note that

$$\begin{aligned}\omega((p,q),(p',q')) = pq' - p'q &= \Big(\sum_{i=1}^{n} dp_i \wedge dq_i\Big)((p,q),(p',q')) \\ &= d\Big(\sum_{i=1}^{n} p_i dq_i\Big)((p,q),(p',q')).\end{aligned}$$

This leads us to think that we should pose ω to be exact. An irremediable consequence of exactness would then be the inexistence of closed symplectic manifolds as the following proposition shows:

Proposition 1.2.3 ([AD14, Prop. 5.2.1.]). *There do not exist any non-degenerate and exact 2-forms on a closed manifold.*

Proof. As noted above, the wedge $\omega^{\wedge n}$ is a volume form. By Stoke's Theorem, volume forms on closed manifolds cannot be exact. But if ω is exact, one can see by induction that $\omega^{\wedge n}$ is exact, which leads to a contradiction. □

We find consolation in the fact that we only wanted to locally emulate computations in $\mathbb{R}^n \times \mathbb{R}^n$. A condition of 'local exactness' is given by the property of closure. Indeed, let ω be a closed 2-form and let $x \in M$. As manifolds are locally-euclidean, we can assume that there exists a neighbourhood U of x such that U is contractible. But then $H^2_{\mathrm{dR}}(U) = 0$ and thus since $d\omega_{|U} = 0$ we have that $d\omega_{|U}$ is exact. Combining all of the above requirements on ω leads us to:

Definition 1.2.4. *A **symplectic manifold** is a pair (M, ω), where M is a manifold and ω is a non-degenerate closed 2-form. Such a form ω is called symplectic.*

Example 1.2.5.

- $\mathbb{R}^n \times \mathbb{R}^n$ with the form $\sum dp_i \wedge dq_i$ is a symplectic manifold.

- Any orientable surface S can be seen as a symplectic manifold. Indeed, any 2-form is closed as $\Omega^3(S) = 0$. Since a volume form of an oriented surface is a non-degenerate 2-form, it provides a symplectic form.

• On the unit sphere S^2, whose tangent space at a point x is a plane orthogonal to the unit vector v, we can define the symplectic form

$$\omega_x(v, w) = x \cdot (v \times w).$$

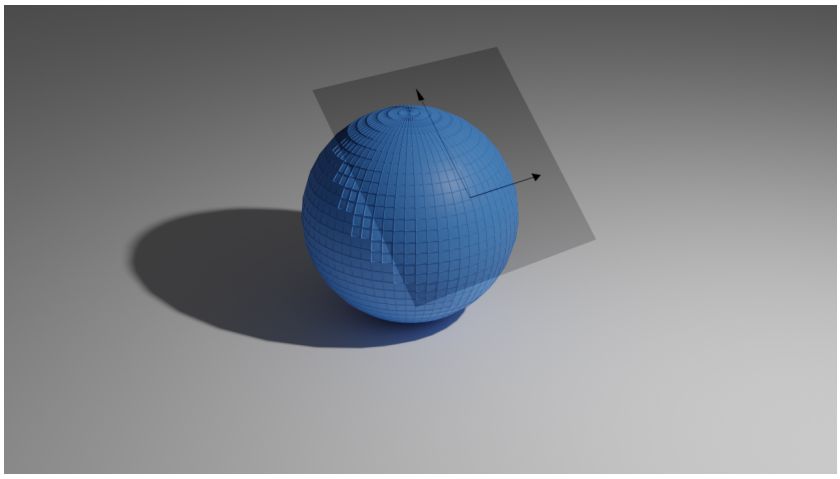


Figure I.1: Symplectic form on the unit sphere

• The most important example of a symplectic manifold that we will encounter is the cotangent bundle with projection

$$\pi : T^*M \to M.$$

We define the tautological 1-form on T^*M by

$$\lambda_{(p,q)}(\xi) = p_q(D\pi(q, p)[\xi]) \quad \text{for } \xi \in T_{(p,q)}T^*M.$$

The canonical symplectic form on T^*M is then defined by $\omega_0 = d\lambda$. In local coordinates, we have the expressions

$$\lambda = \sum p_i dq_i \text{ and } \omega_0 = \sum dp_i \wedge dq_i.$$

As is tradition in mathematics, we give maps that preserve our newly defined structure a name.

Definition 1.2.6. *A diffeomorphism f between two symplectic manifolds (M, ω) and (M', ω') is called a* ***symplectomorphism*** *if*

$$f^*\omega' = \omega.$$

1.3 Hamiltonian Vector Fields

Let (M, ω) be a symplectic manifold and let $H : M \to \mathbb{R}$ be a smooth function called the Hamiltonian. Its differential dH is a 1-form and we can use the symplectic structure to associate a vector field X_H to H by the relation:

$$\iota_{X_H}\omega = -dH.$$

By non-degeneracy, this relation defines X_H uniquely and we will call X_H the *Hamiltonian vector field* associated to H. The above construction is in close analogy to the definition of the (negative) pseudo-gradient vector field in Morse theory.

We define the Hamiltonian system associated to X_H to be the following differential system:

$$\dot{x}(t) = X_H(x(t)).$$

Note that the value of H is constant along the trajectories:

$$\begin{aligned}\frac{d}{dt}H(x(t)) = dH_{|_{x(t)}}(\dot{x}(t)) &= dH(X_H)(x(t)) \\ &= -\omega_{|_{x(t)}}(X_H(x(t)), X_H(x(t))) = 0\end{aligned}$$

since $\omega_{|_{x(t)}}$ is alternating for all t. Going back to the previously described Hamiltonian formalism of classical mechanics, where H was the total energy of a physical system, the above observation translates to the energy being conserved if we let the system evolve in time by the equations of motion given to us by the Hamiltonian equations (1.1.1).

This link to classical mechanics further manifests in the fact that we are able to recover Hamilton's equations in the following way: take $\mathbb{R}^{2n}$ equipped with the symplectic form $\omega = \sum dq_i \wedge dp_i$. Then

$$\begin{aligned}\omega_{|_{x(t)}}\Big(\frac{\partial}{\partial q_i}_{|x(t)}, \dot{x}(t)\Big) &= \Big(\sum dq_{j_{|_{x(t)}}} \wedge dp_{j_{|_{x(t)}}}\Big)\Big(\frac{\partial}{\partial q_i}_{|_{x(t)}}, \dot{x}(t)\Big) \\ &= dp_{i_{|_{x(t)}}}(\dot{x}(t)) = \dot{x}(t)(p_i) = \dot{p}_i(t)\end{aligned}$$

But also

$$\omega_{|_{x(t)}}\Big(\frac{\partial}{\partial q_i}_{|x(t)}, \dot{x}(t)\Big) = -dH\Big(\frac{\partial}{\partial q_i}_{|x(t)}\Big) = -\frac{\partial H}{\partial q_i}(x(t)).$$

And similarly

$$\dot{q}_i(t) = \frac{\partial H}{\partial p_i}(x(t)).$$

Thus we have recovered equations (1.1.1). We will often consider a *time-dependent* Hamiltonian, i.e.

$$H : W \times \mathbb{R} \to \mathbb{R}.$$

The Hamiltonian vector fields are thus given as:

$$X_t := X_{H_t} \quad \text{if we set} \quad H(x,t) = H_t(x).$$

The associated dynamical system is then no longer autonomous, it is the system

$$\dot{x}(t) = X_t(x(t)).$$

Denoting the flow of X_H by Φ_H^t we have the following important observation:

Proposition 1.3.1 ([AD14, Prop. 5.4.2.]). *The time-t flow of a Hamiltonian vector field is a diffeomorphism that preserves the symplectic form. It is called a Hamiltonian diffeomorphism.*

Proof. We have

$$\begin{aligned}\frac{d}{dt}((\Phi_H^t)^*\omega) &= (\Phi_H^t)^*\mathcal{L}_{X_H}\omega \\ &= (\Phi_H^t)(di_{X_H}\omega) \\ &= (\Phi_H^t)(-ddH) = 0.\end{aligned}$$

□

We will be interested in the set of 1-periodic solutions that satisfy the *non-degenarcy condition* that 1 is not an eigenvalue of $D\Phi_H^1(x(0))$.

1.4 Almost Complex Structures

Given a symplectic manifold (M, ω), it is possible to endow it with a structure that imitates multiplication by i in $\mathbb{C}$. Let (V, ω) be a symplectic vector space of dimension $2n$. We define an almost complex structure to be an endomorphism J on V such that $J^2 = -\mathrm{id}$. We say that J is compatible with ω if

$$\omega(Jv, Jw) = \omega(v, w)$$

and if

$$\begin{aligned} g : V \times V &\to \mathbb{R} \\ (v, w) &\mapsto \omega(v, Jw) \end{aligned}$$

is an inner product on V. Almost complex structures always exist on symplectic vector spaces as we can identify V with $(\mathbb{R}^{2n}, \omega_0)$ and then $\mathbb{R}^{2n}$ with $\mathbb{C}^n$. Multiplication by i then fulfils the requirements.

An almost complex structure J on a symplectic manifold (M, ω) is an endomorphism of the tangent bundle TM such that $J^2 = -\mathrm{id}$. We say that J is compatible with ω if J is an isometry of ω, and if the bilinear form $\omega(X, JY)$ is positive definite at every point, i.e. if J_x is compatible with ω_x at every point $x \in M$ (in the sense of vector spaces).

Example 1.4.1 ([AD14, Ex. 5.5.2]). Let S be an oriented surface endowed with a Riemannnian metric. At every point $x \in S$, the tangent space T_xS is of dimension 2. Thus a rotation of $\frac{\pi}{2}$ in each tangent space is well-defined and the family of such rotations constitutes an almost complex structure.

2 Symplectic Cohomology of the Cotangent Bundle

To start off, we give a broad outline of the construction of Floer cohomology that is to follow in the subsequent sections. For this overview, we took inspiration from the excellent introductions given in [Sch93] and [Wen16].

The driving force in the development of Floer homology was the quest for a proof of Arnold's conjecture originating from celestial mechanics of which one version states as follows:

CONJECTURE 2.0.1 (Arnold). *Let (M, ω) be a compact symplectic manifold and $H_t : M \to \mathbb{R}$ a smooth time-dependent 1-periodic Hamiltonian. Suppose that the 1-periodic orbits of X_{H_t} are non-degenerate. Then the number of such orbits is at least the sum of the Betti number of M.*

The Arnold conjecture as stated here has been proved in full generality. The following historical account is taken from [Sal97]. The preceding conjecture was first shown for Riemann surfaces by Eliashberg in [Eli79] and later on confirmed by Conley and Zehnder for the $2n$-torus in [CZ83]. Gromov in [Gro85] proved the existence of at least one fixed point under the assumption $\pi_2(M) = 0$. The breakthrough came in the form of the seminal papers [Flo88a; Flo88b; Flo88c; Flo89b], wherein Floer was able to combine the variational approach of Conley and Zehnder with the elliptic techniques of Gromov and the Morse-Smale-Witten complex to develop a novel infinite dimensional approach to Morse theory, now called Floer homology. This work culminated in the paper [Flo89a], where Floer was able to prove Arnold's conjecture for monotone symplectic manifolds. Subsequently, these ideas were extended by Hofer and Salamon and, independently, by Ono for the weakly monotone case. The final leap to the general case was then independently performed by Fukaya and Ono, by Liu and Tian, and Hofer and Salamon. We refer to [Sal97] for the precise references to these works.

In many ways, (Hamiltonian) Floer homology was inspired by Morse homology. This analogy has its origins in the classical result of Morse theory that the closed geodesics on a manifold M are equal to the critical points of the so-called **energy functional**:

$$\begin{aligned} E : \mathcal{L}M &\to \mathbb{R} \\ \gamma &\mapsto \int_{S^1} |\dot{\gamma}(t)|^2 dt \end{aligned}$$

defined on some loop space $\mathcal{L}M$ satisfying certain conditions that we won't get into at this point. For the sake of exposition, assume that (M, ω) is symplectically aspherical, i.e. that $[\omega]_{|\pi_2(M)} = 0$, that the loops we consider are contractible, as well as the existence of a (time-dependent) compatible almost complex structure $J(t, x)$. We will see later on how we can remove or weaken these conditions in the case of the cotangent bundle due to the fact that its symplectic form is exact. A

key observation is then that the critical points of the **action functional**:

$$\begin{aligned}\mathcal{A}_H : C^\infty_{\text{contr}} &\to \mathbb{R} \\ x &\mapsto -\int_{D^2} \tilde{x}^*\omega + \int_{S^1} H_t(x(t))dt,\end{aligned}$$

where $\tilde{x} : D^2 \to M$ is a smooth map with $\tilde{x}_{|\partial D^2} = x$, are exactly the contractible 1-periodic orbits of X_{H_t}. This enables us to restate the question of existence of solutions of the differential equation

$$X_{H_t}(x(t)) = \dot{x}(t)$$

in terms of critical points of $\mathcal{A}_H$. The set of time-1 Hamiltonian orbits will be denoted by $\mathcal{O}(H)$. In his work, Floer developed a novel infinite dimensional Morse theory for $\mathcal{A}_H$, where this functional plays the role of a Morse function. However a naive application of Morse homology fails in its tracks as the following shows. In order to define a gradient, identify the tangent space at a loop x with vector fields along x, $\Gamma(x^*TM)$. One then introduces an L^2-inner product on $\Gamma(x^*TM)$ and the gradient of $\mathcal{A}_H$ with respect to this inner product turns out to be

$$\nabla_J \mathcal{A}_H(x)(t) = -J(t,x)(\dot{x} - X_t(t,x)).$$

At this point the standard Morse theoretic approach breaks as the subspaces on which the Hessian is maximally positive or negative definite are of infinite dimension. In addition, the gradient doesn't even lead to well-defined flowed lines, it is unregularised.

Floer's new insight was that even though the standard conditions were not met, one could still define a relative version of Morse homology. This novel approach consists of restricting the gradient flow to flows between two fixed critical points x^- and x^+ of $\mathcal{A}_H$ of a (to be defined) relative index that, loosely speaking, measures the codimension of the unstable manifold of x^- with respect to the unstable manifold of x^+.

Having identified the gradient $\nabla_J \mathcal{A}_H$, the negative gradient equation $\partial_s u + \nabla_J \mathcal{A}_H(u) = 0$ transforms into the (elliptic) PDE:

$$\partial_s u + J(t,u)(\partial_t u - X_t(t,u)) = 0. \tag{2.0.1}$$

This is the so-called Floer equation and its solutions will be called trajectories. Define $\widetilde{\mathcal{M}}(x^-, x^+)$ to be the set of solutions u of (2.0.1) such that $\lim_{s\to\pm\infty} u(s) = x^\pm$. As Arnold's conjecture only concerns itself with nondegenerate solutions, we are going to make the following standing assumption on the Hamiltonian H:

(H0) every solution x of (2.0.1) is nondegenerate, meaning that 1 is not an eigenvalue of $D\Phi^1_H(x(0))$, where Φ^t_H is the flow of the Hamiltonian vector field.

Denote by $CF_k(H)$ the free abelian group generated by the critical points of $\mathcal{A}_H$ of index k. Floer then showed that there is a well-defined map

$$\partial_k : CF_k(H) \to CF_{k-1}(H)$$

that is defined on a generator x of index k as

$$\partial_k x = \sum_{\substack{y \in \{\text{crit. p. of } \mathcal{A}_H\} \\ \text{index}(y)=k-1}} n(x,y)y,$$

where $n(x,y) \in \mathbb{Z}$ is a number depending on the orientations of the trajectories linking to x. In addition, this map satisfies $\partial_{k-1} \circ \partial_k = 0$ such that $(CF(H), \partial)$ forms a chain complex. The resulting homology is then called Floer homology.

For our exposition however we will concern ourselves with open manifolds, in particular cotangent bundles. This leads to a number of technical issues that need to be resolved. The main difference to Hamiltonian Floer theory on a closed symplectic manifold are the following:

- For closed manifolds, the Floer complex is defined for a generic Hamiltonian and generic compatible almost complex structure. The (co)homology of the complex can then be shown to be independent of these choices. This is not true on the cotangent bundle, and therefore one has to impose additional assumptions on the Hamiltonian in order to rectify this.
- One usually looks at contractible orbits under the assumption that the first Chern class vanishes. For non-orientable manifolds, the first Chern class does not have to vanish (ex: $\mathbb{R}\mathrm{P}^2$). So we must also concern ourselves with non-contractible loops.

In the subsequent chapter will take special care to specify the additional assumptions needed on the Hamiltonians and almost complex structures to define Floer cohomology on the cotangent bundle. In addition, we will attach a local system to the Floer chain complexes in view of being able to state Viterbo's Theorem in full generality. In order to focus on the subtle differences between the closed and the open case, we will assume some familiarity with classical Floer homology on closed manifolds as for example presented in [AD14].

2.1 The Action Functional

Let $\mathcal{L}T^*M$ be the space of loops $x : \mathbb{R}/\mathbb{Z} \to T^*M$ of class C^∞. It is possible to endow $\mathcal{L}T^*M$ with the structure of a Banach manifold. Formally, a tangent vector at $x \in \mathcal{L}T^*M$ is given by an equivalence class of curves passing through x. Consider

$$\begin{aligned} u : \mathbb{R} &\to \mathcal{L}T^*M \\ s &\mapsto u(s) \quad \text{with } u(0) = x. \end{aligned}$$

We will use the notation $u(s,t) := u(s)(t)$. We have

$$\frac{\partial}{\partial s}\Big|_{s=0} u(s,t) \in T_{x(t)}T^*M,$$

so that is natural to view a tangent vector of $\mathcal{L}T^*M$ at x as a vector field on T^*M along x, or in symbols $Y \in \Gamma(x^*TT^*M)$. Note that Y is automatically 1-periodic.

The explicit Banach manifold structure is then for example given in [AD14, Thm. 6.8.1.]

Recall the previous setting — we are given a 1-periodic Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times T^*M \to \mathbb{R}$ that determines the Hamiltonian vector field X_H characterised by the equation:

$$\omega(X_H(t,x),\zeta) = -dH(t,x)[\zeta] \quad \forall \zeta \in T_xT^*M.$$

Consider the action functional

$$\mathcal{A}_H(x) := \int x^*(-\lambda + Hdt) = \int_0^1 (-\lambda(\dot{x}) + H(t,x))dt.$$

It is smooth on $\mathcal{L}T^*M$.

Example 2.1.1 ([AD14, Example 6.3.1.]). Consider the case $T^*M = \mathbb{R}^{2n}$ with the tautological one-form $\lambda = \sum p_i dq_i$. Then

$$\int x^*(\lambda) = \sum \int_0^1 p_i dq_i = \sum \int_0^1 p_i q_i dt = \int_0^1 \langle \vec{p}, \dot{\vec{q}} \rangle dt.$$

Thus we recover the physicist's action integral (1.1.2).

Having identified the tangent vectors of the loop space, we are able to give an expression for the differential:

Proposition 2.1.2. *For $Y \in T_x\mathcal{L}T^*M$, the differential of $\mathcal{A}_H$ is given by:*

$$d\mathcal{A}_H(x)[Y] = \int_0^1 \omega_{x(t)}(Y(t), -\dot{x}(t) + X_H(t,x))dt.$$

In addition, we have that a loop x is a critical point of $\mathcal{A}_H$ if and only if $t \mapsto x(t)$ is a periodic solution of the Hamiltonian system $\dot{x} = X_t(x)$.

Proof. Accepting the form of $d\mathcal{A}_H(x)$ for the time being, we have that

$$dA_H(x)[Y] = 0 \ \forall Y \iff \omega(-\dot{x}(t) + X_t(x), Y) = 0 \ \forall Y.$$

The nondegeneracy of ω implies that the above is true if and only if $\dot{x}(t) = X_t(x)$, i.e. x is a solution of $\dot{x} = X_t(x)$.

In contrary, computing the differential is a bit trickier. Given a tangent vector Y at x, we can, by the definition of the tangent vectors on $\mathcal{L}T^*M$, find a path of loops $\tilde{x}(s) \in \mathcal{L}T^*M$ such that

$$\begin{cases} \tilde{x}(0,t) = x(t), \\ \frac{\partial \tilde{x}}{\partial s}(0,t) = Y(t). \end{cases}$$

We have $d\mathcal{A}_H(x)[Y] = \frac{\partial}{\partial s}_{|_{s=0}} \mathcal{A}_H(\tilde{x})$. By definition, we have:

$$\mathcal{A}_H(\tilde{x}(s)) = -\int_{S^1} \tilde{x}^*\lambda + \int_0^1 H_t(\tilde{x}(s,t))dt.$$

Differentiating the first term:

$$\begin{aligned}\int_{S^1} \frac{d}{ds}\Big|_{s=0} \tilde{x}^*\lambda &= \int_{S^1} x^*(\mathcal{L}_Y\lambda) = \int_{S^1} x^*(di_Y\lambda) + \int_{S^1} x^*(i_Y d\lambda) \\ &= \int_{S^1} x^* d(\lambda(Y)) + \int_{S^1} x^*(i_Y\omega) \\ &= \int_0^1 d(\lambda(Y))dt + \int_0^1 \omega(Y,\dot{x})dt \\ &= \underbrace{\lambda(Y(1)) - \lambda(Y(0))}_{=0 \text{ by periodicity of Y}} + \int_0^1 \omega_{x(t)}(Y(t),\dot{x}(t))dt.\end{aligned}$$

Differentiating the second term turns out to be easier:

$$\begin{aligned}\int_0^1 \frac{\partial}{\partial s}\Big|_{s=0} H_t(\tilde{x}(s,t))dt &= \int_0^1 (dH_t)_{\tilde{x}(0,t)}(Y(t))dt \\ &= \int_0^1 \omega_{x(t)}(Y(t), X_t(x(t))dt.\end{aligned}$$

Putting these together yields

$$d\mathcal{A}_H(x)[Y] = \int_0^1 \omega_{x(t)}(Y(t), -\dot{x}(t) + X_H(t,x))dt.$$

□

As just seen in Proposition 2.1.2, the critical points of $\mathcal{A}_H$ are the elements of $\mathcal{O}(H)$. It was Floer's idea to use the L^2-gradient flow equations for $\mathcal{A}_H$ in order to build a chain complex for his new homology theory.

Let $J(t,x)$ be a smooth almost complex structure on T^*M that is 1-periodic in t, and compatible with ω at every time step, i.e.

$$\langle \zeta_1, \zeta_2 \rangle_{J_t} := \omega(\zeta_1, J(t,x)\zeta_2) \quad \zeta_1, \zeta_2 \in T_{x(t)}T^*M, \ x(t) \in T^*M \tag{2.1.1}$$

is a loop of Riemannian metrics on T^*M. Consequently $J(t,x)$ provides us with a complex structure along every loop $x \in \mathcal{L}T^*M$. Fix $x \in \mathcal{L}T^*M$, then given $Y, Z \in T_x\mathcal{L}T^*M$, we have an induced metric on $T_x\mathcal{L}T^*M$ given by:

$$\langle Y, Z \rangle := \int_0^1 \langle Y(t), Z(t) \rangle_{J_t} dt = \int_0^1 \omega_{x(t)}(Y(t), J(t,x)Z(t))dt.$$

The gradient vector field of $\mathcal{A}_H$, denoted by $\nabla_J\mathcal{A}_H$, is then defined by the equation

$$\begin{aligned}\langle \nabla_J\mathcal{A}_H(x), Y \rangle_x &= \int_0^1 \omega_{x(t)}(\nabla_J\mathcal{A}_H(x)(t), J(t,x)Y(t))dt \\ &\overset{!}{=} d\mathcal{A}_H(x)[Y] = \int_0^1 \omega_{x(t)}(Y, -\dot{x} + X_H(t,x))dt.\end{aligned}$$

Since

$$\omega_{x(t)}(Y, -\dot{x} + X_H(t,x)) = \omega_{x(t)}(-J(t,x)(-\dot{x} + X_H(t,x)), J(t,x)Y(t)),$$

we have that

$$\nabla_J \mathcal{A}_H(x)(t) = -J(t,x)(-\dot{x} + X_H(t,x)).$$

The trajectories of the vector field $\nabla_J \mathcal{A}_H$ are therefore the solutions

$$\begin{aligned} u : \mathbb{R} &\to \mathcal{L}T^*M \\ s &\mapsto u(s) \end{aligned}$$

of the *negative gradient equation*

$$\frac{\partial u}{\partial s} = J(t, u(s,t))\bigg(- \frac{\partial u}{\partial t} + X_H(t, u(s,t)\bigg),$$

or written more compactly

$$\partial_s u + J(t,u)(\partial_t u - X_H(t,u)) = 0 \tag{2.1.2}$$

with $u \in C^\infty(\mathbb{R} \times (\mathbb{R}/\mathbb{Z}), T^*M)$.

2.2 Solutions of the Floer Equation — the Space $\mathcal{M}(x_0, x_1)$

Given two loops x_0 and x_1 that are critical points of the action functional $\mathcal{A}_H$, we now examine the space of solutions of the Floer equation (2.1.2). Define

$$\widetilde{\mathcal{M}}(x_0, x_1) := \{\ u \text{ solution of (2.1.2)} \mid \lim_{s\to-\infty} u(s,t) = x_0(t) \text{ and } \lim_{s\to-\infty} u(s,t) = x_1(t)\}.$$

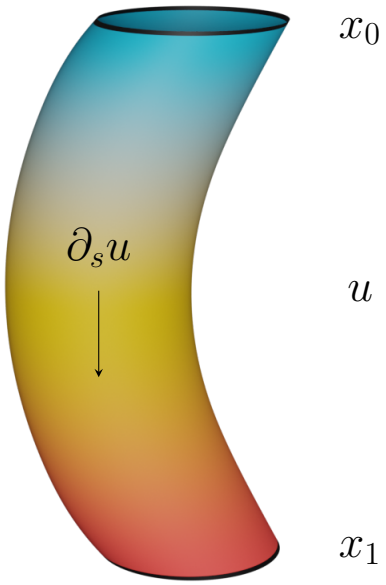


Figure I.2: A holomorphic cylinder u linking two orbits x_0 and x_1

We have the following interesting result:

PROPOSITION 2.2.1. *The elements of $\mathcal{O}(H)$ are the stationary solutions of (2.1.2). Furthermore, if u is any other solution of (2.1.2) and $s \in \mathbb{R}$ is such that $u(s) \notin \mathcal{O}(H)$, then $\mathcal{A}_H(u(s+\varepsilon)) - \mathcal{A}_H(u(s)) < 0$ for all $\varepsilon > 0$.*

An immediate consequence of this proposition is the following:

COROLLARY 2.2.2. *Let $x, x_0, x_1 \in \mathcal{O}(H)$, then*

$$\begin{aligned} \widetilde{\mathcal{M}}(x,x) &= \{x\}, \\ \widetilde{\mathcal{M}}(x_0,x_1) &= \varnothing \quad \text{if } \mathcal{A}_H(x_0) \leq \mathcal{A}_H(x_1) \text{ and } x_0 \neq x_1. \end{aligned}$$

Proof of Proposition 2.2.1. If u does not depend on s, i.e. $u(s,t) = x(t)$ for some $x \in \mathcal{L}T^*M$, then (2.1.2) becomes

$$J(t,x)\dot{x} = J(t,x)X_H \implies \dot{x} = X_H,$$

but in Proposition 2.1.2 we already identified these solutions as being the stationary points of the action functional.

On the other hand, if $u(s) \notin \mathcal{O}(H)$ then we have:

$$\begin{aligned} \frac{\partial}{\partial s}\mathcal{A}_H(u(s)) &= d\mathcal{A}_H(u(s))\left[\frac{\partial u}{\partial s}\right] \\ &= \int_0^1 \omega\left(\frac{\partial u}{\partial s}, -\frac{\partial u}{\partial t} + X_H(t,u(s))\right)dt \\ &= \int_0^1 \omega\Big(- J\left(\frac{\partial u}{\partial t} - X_H\right), -\frac{\partial u}{\partial t} + X_H\Big)dt \\ &= -\int_0^1 \underbrace{\left\langle \frac{\partial u}{\partial t} - X_H, \frac{\partial u}{\partial t} - X_H \right\rangle_{J_t}}_{>0 \text{ as } u(s) \notin \mathcal{O}_H} dt < 0. \end{aligned}$$

□

We will now briefly describe the functional analytic setting that allows us to see $\widetilde{\mathcal{M}}(x_0,x_1)$ as the set of zeros of a smooth section of a Banach bundle. Let us fix a number $r > 2$, and recall that maps on two-dimensional domains of Sobolev class $W^{1,r}$ are Hölder continuous by the Sobolev embedding.

Define $\mathcal{P}^{1,r}(x_0,x_1)$ to be the set of all maps $u : \mathbb{R} \times S^1 \to T^*M$ of Sobolev class $W^{1,r}_{\text{loc}}$ such that there exists an $s_0 \geq 0$ for which

$$u(s,t) = \begin{cases} \exp_{x_0(t)}(\zeta_0(s,t)) & \forall s \leq -s_0 \\ \exp_{x_1(t)}(\zeta_1(s,t)) & \forall s \geq s_0, \end{cases}$$

where ζ_0 and ζ_1 are $W^{1,r}$ section of the pullback bundles $x_0^*(TT^*M)$ and $x_1^*(TT^*M)$ respectively. Here exp denotes the exponential map with respect to some metric on T^*M, and the space of $W^{1,r}$ sections is also defined in terms of this metric. Then $\mathcal{P}^{1,r}(x_0, x_1)$ can be given the structure of a smooth Banach manifold, see for example [AD14, Sec. 8.2]. Due to the fact that we are not restricting ourselves to contractible loops however, one has to use a different method to trivialise the pullback bundles than the one used by Audin and Damian, for example [AS06, Lem. 1.1]. The tangent space at $u \in \mathcal{P}^{1,r}(x_0, x_1)$ can then be identified with the space of $W^{1,r}$ sections of $u^*(TT^*M)$.

Let $\mathcal{L}^r(x^-, x^+)$ be the fibre bundle over $\mathcal{P}^{1,r}(x^-, x^+)$ whose fibre at u is the space of L^r sections of $u^*(TT^*M)$. Define the *Floer map*

$$\begin{aligned}\partial_{J,H} : \mathcal{P}^{1,r}(x^-, x^+) &\to \mathcal{L}^r(x^-, x^+) \\ u &\mapsto \frac{\partial u}{\partial s} + \nabla_J \mathcal{A}_H(u) = \frac{\partial u}{\partial s} + J(t,u)\left(\frac{\partial u}{\partial t} - X_H(t,u)\right).\end{aligned}$$

The fact that $\partial_{J,H}$ is indeed a well-defined map is for instance shown in [AD14, Sec. 13.3]. The space $\widetilde{\mathcal{M}}(x_0, x_1)$ presents a natural $(\mathbb{R}, +)$ action defined by:

$$(u \cdot \sigma)(s,t) = u(s + \sigma, t) \quad \forall \sigma \in \mathbb{R}.$$

Definition 2.2.3. *Let $x_0, x_1 \in \mathcal{O}(H)$. The moduli space $\mathcal{M}(x_0, x_1)$ is the quotient of $\widetilde{\mathcal{M}}(x_0, x_1)$ by the above defined action of $\mathbb{R}$.*

Next we will analyse the fibrewise derivative of the Floer map at a solution $u \in \mathcal{M}(x_0, x_1)$. It is defined as follows:

$$D_u := D_f \partial_{J,H}(u) = \mathrm{pr}_{E_u} \circ D\partial_{J,H}(u), \tag{2.2.1}$$

where pr_{E_u} is the projection onto the tangent space of the fibre (which can be canonically identified with the fibre itself as it is a vector space). In [Flo88a], Floer showed that this map is a Fredholm operator

$$D_u : W^{1,r}(\mathbb{R} \times S^1; u^*(TT^*M)) \to L^r(\mathbb{R} \times S^1; u^*(TT^*M)).$$

We have the following proposition that will come in handy later on:

Proposition 2.2.4. *Let $u \in \widetilde{\mathcal{M}}(x_0, x_1)$ for $x_0, x_1 \in \mathcal{O}(H)$ such that $x_0 \neq x_1$. There exists a non-trivial element of the kernel of D_u given by $\partial_s u$.*

Proof. As translation presents an action on the space of solutions to Floer's equation, we have that $\partial_{J,H}(u \cdot s) = 0$ for all $s \in \mathbb{R}$. Thus

$$0 = \frac{d}{ds}\partial_{J,H}(u \cdot s) = D_u\left(\frac{\partial u}{\partial s}\right).$$

□

2.3 Compactness of the Moduli Space

Up to this point, almost none of our arguments were tailored to the fact that we work with open manifolds. This changes however with compactness, since without additional assumptions, sequences of trajectories don't need to converge as we might not be able to control the behaviour at infinity. We thus impose additional assumptions on the Hamiltonian and almost complex structure. From here on and for the rest of the section, we will match [Abo14, Sec.1] closely.

Let us now fix a metric on M, and hence also an induced metric on T^*M. We write

$$\rho(q,p) = \langle p,p\rangle^{\frac{1}{2}}$$

for the norm of the covector p. One can think of ρ as being a radial coordinate on T^*M. For each positive real number ρ, we obtain a disc bundle

$$D^*_\rho M \subset T^*M$$

consisting of those points (p,q) such that $\langle p,p\rangle^{\frac{1}{2}} \leq \rho$. The boundary of $D^*_\rho M$ is the sphere bundle, which we denote by $S^*_\rho M$. When $\rho = 1$, we omit the subscript from the notation.

Rescaling the fibres allows us to identify the complement of D^*M with the product of S^*M with a ray; we obtain a decomposition

$$T^*M = D^*M \cup_{S^*M} S^*M \times [1,+\infty)$$

into the disc bundle and a conical end.

In order to tackle the problem of defining Floer homology on the cotangent bundle, we will restrict ourselves to a special type of Hamiltonians — ones with controlled behaviour at infinity.

Definition 2.3.1. *Let b be a real number. A Hamiltonian H is linear of slope b if*

$$H|_{S^*M\times[1,+\infty)} \equiv b\cdot\rho.$$

We define a preorder on the set of linear Hamiltonians:

$$H \leq K \quad \text{if the slope of } H \text{ is less than or equals that of } K.$$

Unless otherwise mentioned, all Hamiltonians considered from now on will be linear. In addition, we impose certain conditions on the complex structure.

Definition 2.3.2. *A compatible almost complex structure J is said to be **convex** near $S^*_\rho M$ if the restriction to a neighbourhood of this hypersurface satisfies*

$$d\rho \circ J = -e^f \lambda$$

for some smooth function f, and for λ the tautological 1-form.

As a final assumption, we will add that the manifolds M, whose cotangent bundles we shall consider, are closed.

We now introduce the useful notion of the energy of a Floer trajectory that will help us show that solutions to the Floer equation cannot escape to infinity. From now on, we will sometimes denote the cylinder $S^1 \times \mathbb{R}$ by Z.

DEFINITION 2.3.3. *We define the* ***energy*** *of a Floer trajectory to be the integral over the cylinder:*

$$E(u) := \frac{1}{2} \int_{-\infty}^{+\infty} \int_{S^1} \left(|\partial_s u|^2 + |\partial_t u - X_t(u)|^2 \right) dt \, ds,$$

where the norms are induced by the family of metrics of (2.1.1), i.e. $|\partial_s u|^2 = \omega(\partial_s u, J_t \partial_s u)$.

REMARK 2.3.4. In his monograph, Abouzaid introduces a tensorial point of view to express the energy [Abo14, Eq. 1.3.11]. For this he uses that the total differential at a solution u:

$$Du(s,t) : T_{(s,t)}Z \to T_{u(s,t)}T^*M$$

can be seen as a section $Du \in \Gamma(\mathrm{Hom}(TZ, u^*TT^*M))$. We then have the following chain of isomorphisms

$$\begin{aligned} \Gamma(\mathrm{Hom}(TZ, u^*TT^*M)) &\cong \mathrm{Hom}(\Gamma(TZ), \Gamma(u^*(TT^*M))) \\ &\cong \mathrm{Hom}(\mathfrak{X}(Z), \Gamma_u(TT^*M)) \\ &\cong \mathfrak{X}(Z)^* \otimes \Gamma_u(TT^*M) \\ &\cong \Omega^1(Z) \otimes \Gamma_u(TT^*M). \end{aligned}$$

Using this isomorphism, we see that the total differential splits as $Du = \partial_s u \otimes ds + \partial_t u \otimes dt$ and we define the *local energy*:

$$\begin{aligned} ||Du - X_{H_t} \otimes dt||^2 &= ||\partial_s u \otimes ds + (\partial_t u - X_{H_t}) \otimes dt||^2 \\ &:= |\partial_s u|^2 + |\partial_t u - X_{H_t}|^2 \end{aligned}$$

in analogy to the euclidean norm. The energy of a Floer trajectory can then be rewritten as

$$E(u) = \int_Z ||Du - X_{H_t} \otimes dt||^2 ds \wedge dt.$$

Given $a < b$, we have the following result:

LEMMA 2.3.5 ([Abo14, Lem. 1.3.11.]). *The energy of the restriction of a solution u to the open cylinder $(a,b) \times S^1$ is given by*

$$E(u|_{(a,b)\times S^1}) = \mathcal{A}_H(u(a,\cdot) - \mathcal{A}_H(u(b,\cdot).$$

Proof. Let u be a solution to the Floer equation. Then:

$$\begin{aligned}
E(u|_{(a,b)\times S^1}) &= \frac{1}{2}\int_{(a,b)\times S^1} (|\partial_s u|^2 + |\partial_t u - X_t(u)|^2) ds \wedge dt \\
&\overset{\spadesuit}{=} \int_{(a,b)\times S^1} |\partial_s u|^2 ds \wedge dt \\
&= \int_{(a,b)\times S^1} \omega(\partial_s u, J_t \partial_s u) ds \wedge dt \\
&= \int_{(a,b)\times S^1} \omega(\partial_s u, \partial_t u - X_t) ds \wedge dt \\
&\overset{\heartsuit}{=} \int_{(a,b)\times S^1} u^*(\omega) - u^*(dH_t) \wedge dt \qquad (2.3.1) \\
&= \int_{(a,b)\times S^1} d(u^*(\lambda)) - d(u^*(H_t)dt) \\
&\overset{\blacklozenge}{=} -\int_{\{a\}\times S^1} u^*(\lambda) - u^*(H_t)dt + \int_{\{b\}\times S^1} u^*(\lambda) - u^*(H_t)dt \\
&= \mathcal{A}_H(u(a,\cdot)) - \mathcal{A}_H(u(b,\cdot)),
\end{aligned}$$

where ($\spadesuit$) used the fact that u is a solution to Floer's equation, ($\blacklozenge$) used Stoke's Theorem and ($\heartsuit$) follows from:

$$\begin{aligned}
(u^*(dH_t) \wedge dt)\Big[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\Big] &= dH_t\Big(\frac{\partial u}{\partial s}\Big) \cdot \underbrace{dt\Big[\frac{\partial}{\partial t}\Big]}_{=1} - dH_t\Big(\frac{\partial u}{\partial t}\Big) \cdot \underbrace{dt\Big[\frac{\partial}{\partial s}\Big]}_{=0} \\
&= \omega(\partial_s u, X_t) ds \wedge dt\Big[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\Big].
\end{aligned}$$

□

An easy consequence of this lemma is the following:

COROLLARY 2.3.6. *The function $\mathcal{A}_H(u(s,\cdot))$ decreases monotonically with s. Furthermore, the moduli space $\mathcal{M}(x_0, x_1)$ is empty unless $\mathcal{A}_H(x_0) > \mathcal{A}_H(x_1)$ or $x_0 = x_1$.*

We are now going to show that if x_0 and x_1 are contained in D^*M, then so are the trajectories of $\mathcal{M}(x_0, x_1)$. We are going to show this assertion by contradiction. Let $u \in \overline{\mathcal{M}}(x_0, x_1)$ be an element that escapes D^*M. Choose a regular value $1+\varepsilon$ of $\rho \circ u$ and define

$$\Sigma := (\rho \circ u)^{-1}([1+\varepsilon, +\infty)).$$

Let v denote $u|_\Sigma$. We then define the *geometric energy* of v by

$$E(v) := \int_\Sigma ||Dv - bX_\rho \otimes dt||^2 ds \wedge dt.$$

Observe that the energy is non-negative. Completely analogously as in the proof of Lemma 2.3.5 we have that

$$E(v) = \int_{\partial\Sigma} v^*(\lambda) - b\rho \circ v \cdot dt.$$

LEMMA 2.3.7 ([Abo14, Lem. 1.3.15.]). *If J_t is convex near S^*M, then the image of every element of $\mathcal{M}(x_0, x_1)$ is contained in D^*M.*

Proof. Assume by contradiction that there is an element u of $\overline{\mathcal{M}}(x_0, x_1)$ whose image intersects the complement of D^*M. By non-negativity of the energy, we have

$$0 < \int_{\partial\Sigma} v^*(\lambda) - b\rho \circ v \cdot dt.$$

We will also show that the opposite is true, thus arriving at a contradiction. Since $\lambda(X_\rho(v(s,t)) = \rho \circ v(s,t)$ we can rewrite the above as

$$\int_{\partial\Sigma} \lambda(Dv - bX_\rho \otimes dt).$$

We have the following chain of equalities

$$\begin{aligned}
\lambda(Dv - bX_\rho \otimes dt) &\overset{\blacklozenge}{=} -\lambda \circ J_t(Dv - bX_\rho \otimes dt) \circ j \\
&\overset{\heartsuit}{=} -e^{-f} d\rho(Dv - bX_\rho \otimes dt) \circ j \\
&\overset{\clubsuit}{=} -e^{-f} d\rho(Dv \circ j + bX_\rho \otimes ds) \\
&= -e^{-f} D(\rho \circ v) \circ j,
\end{aligned}$$

where ($\blacklozenge$) follows from the fact that v is a holomorphic cylinder, ($\heartsuit$) follows by convexity and ($\clubsuit$) follows from the fact that $dt \circ j = ds$. For the last equation, we used that X_ρ is tangent to the level sets of ρ. Recall that a tangent vector ξ to $\partial\Sigma$ is positively oriented if $j\xi$ points inwards. Thus if ξ is a positively oriented tangent vector, we have that $0 \leq D(\rho \circ v)[j\xi]$ since $\rho \circ v$ reaches its global minimum on $\partial\Sigma$. This implies that

$$\int_{\partial\Sigma} -e^{-f} D(\rho \circ v) \circ j \leq 0.$$

We thus conclude that u is contained in a disc bundle of radius $1 + \varepsilon$. Since ε was arbitrary, we have that the image of elements of $\mathcal{M}(x_0, x_1)$ is contained in the disc bundle. □

Thus if we can show that all the element of $\mathcal{O}(H)$ are contained in D^*M, we are able to reduce the question of compactness of the moduli spaces to the closed case. First, we have the following result about the behaviour of solution at the ends:

LEMMA 2.3.8 ([AD14, Thm. 6.5.6.]). *If u is a solution to Floer's equation, then $E(u)$ is finite if and only if there exist orbits x_0 and x_1 such that $u \in \mathcal{M}(x_0, x_1)$.*

□ref.

In order to show compactness, we need the following result:

LEMMA 2.3.9 ([Abo14, Lem. 1.2.9.]). *Let x be an orbit of a linear Hamiltonian H of slope b. If x intersects the conical end, then the loop*

$$\begin{aligned}
\mathbb{R}/b\mathbb{Z} &\to M \\
t &\mapsto \pi(x(t/b))
\end{aligned}$$

is a geodesic parametrised by unit speed.

□ref.

Corollary 2.3.10 ([Abo14, Cor. 1.2.10.]). *If M does not admit any closed geodesics of length b, and H is linear of slope b, then all elements of $\mathcal{O}(H)$ have image contained in the interior of D^*M.*

Now as the set of lengths of geodesics of a closed manifold is a closed set of measure 0, we have that the image of elements of $\mathcal{O}(H)$ is contained in D^*M for a generic choice of Hamiltonian.

Corollary 2.3.11. *For a generic Hamiltonian H, the moduli space $\overline{\mathcal{M}}(x_0, x_1)$ is compact for any pair x_0 and x_1.*

Remark 2.3.12. The proof of the above corollary, which can be found in [AD14, Cor. 9.1.9.] relies on the fact that the cotangent bundle is an *aspherical* symplectic manifold, i.e. that for every C^∞-map $w : S^2 \to W$, we have that

$$\int_{S^2} w^*\omega = 0.$$

This follows immediately from exactness of the symplectic form.

For future constructions, we will need that the number of orbits is invariant under small perturbations. In order to make this explicit, integrate the Hamiltonian vector field X_H to obtain a family of Hamiltonian symplectomorphisms

$$\Phi^t : T^*M \to T^*M \tag{2.3.2}$$

such that $\Phi^t(x(0)) = x(t)$ for every flow line x of X_h. Thus if we restrict to time-1 periodic orbits, we have that $x(0)$ is a fixed point of Φ^1, and thus we obtain a map

$$D\Phi^1(x_0) : T_{x(0)}T^*M \to T_{x(0)}T^*M. \tag{2.3.3}$$

Definition 2.3.13. *A Hamiltonian orbit x is said to be* ***non-degenerate*** *if 1 is not an eigenvalue of $D\Phi^1(x_0)$.*

We have the following genericity result:

Lemma 2.3.14 ([Abo14, Lem. 1.2.13.]). *Let H be a Hamiltonian on T^*M. If $U \subset T^*M$ is an open set, there is a countable intersection $\mathscr{H}(U)$ of open dense subsets in the space of compactly supported smooth functions on $U \times S^1$, such that all orbits of $H + K$ which pass through U are non-degenerate whenever $K \in \mathscr{H}(U)$.*

□ref.

In practice, we will be working with linear Hamiltonians and therefore we can be more precise. As the lengths of the geodesics form a closed set of measure 0, there exists an arbitrary large choice of b such that M does not have any closed geodesics of length b. Thus as an immediate corollary of Lemma 2.3.14, we have:

Corollary 2.3.15 ([Abo14, Cor. 1.2.14.]). *If b is not the length of any geodesic on M, and H is a generic Hamiltonian of slope b, then all element of $\mathcal{O}(H)$ are non-degenerate.*

Another key ingredient for Floer theory is a transversality result that goes back to Floer [Flo88d]. In order to state transversality in our setting, assume that we have fixed a Hamiltonian H such that all orbits are non-degenerate.

THEOREM 2.3.16 ([Abo14, Thm. 1.3.18.]). *There is a dense set $\mathcal{J}_{S^1}^{reg} \subset \mathcal{J}_{S^1}$ such that the following holds whenever $J_t \in \mathcal{J}_{S^1}^{reg}$:*

$$\text{for every pair } (x_0, x_1) \text{ of orbits and every cylinder } u \in \mathcal{M}(x_0; x_1), \quad \text{the operator } D_u \text{ is surjective.} \tag{2.3.4}$$

In this case, $\mathcal{M}(x_0; x_1)$ is a smooth manifold of dimension equal, at every point, to the virtual dimension.

We say that our data (H, J_t) are regular if all orbits are non-degenerate, and Condition 2.3.4 holds. From now on, all such data will be assumed to be regular. We shall be particularly interested in the situation when $\mathcal{M}(x_0; x_1)$ has virtual dimension equal to 0:

DEFINITION 2.3.17. *An element $u \in \mathcal{M}(x_0; x_1)$ is **rigid** if it is regular, and the Fredholm index of D_u is equal to 1.*

2.4 Orientations and Grading

In order to define the differential, Floer gave an orientation to each moduli space which is consistent with the breaking of Floer trajectories. We are however going to follow a different approach as presented in [Abo14]. We will assign to each orbit an *orientation line*, which is a free abelian group of rank 1, and construct a canonical map on orientation lines associated to each rigid Floer trajectory.

2.4.a) The Determinant Line Bundle

Let X and Y be real Banach spaces. Let $\mathrm{Fred}_{\mathbb{R}}(X, Y)$ denote the space of $\mathbb{R}$-linear Fredholm operators. If V is an n-dimensional vector space, we will use the following notation:

$$\bigwedge^{\max} V := \bigwedge^{n} V.$$

Recall that one has the canonical isomorphism $(\bigwedge^{\max} V)^* = \bigwedge^{\max} V^*$. If $\dim V = 0$, then we set $\bigwedge^{\max} V := \mathbb{R}$.

DEFINITION 2.4.1. *Given $T \in \mathrm{Fred}_{\mathbb{R}}(X, Y)$, the **determinant line** of T is the real 1-dimensional vector space*

$$\det(T) := \left(\bigwedge^{max} \ker T\right) \otimes \left(\bigwedge^{max} \operatorname{coker} T\right)^*.$$

The union of these vector spaces defines the set

$$\det(X, Y) := \{(T, v) \mid T \in \mathrm{Fred}_{\mathbb{R}}(X, Y),\ v \in \det(T)\}.$$

We then have the following smooth vector bundle structure:

THEOREM 2.4.2 ([Wen16, Thm. 11.9.]). *There is a smooth vector bundle* $\det(X,Y) \xrightarrow{\pi} \mathrm{Fred}_{\mathbb{R}}(X,Y)$ *of rank 1 such that* $\pi^{-1}(T) = \det(T)$ *for each* $T \in \mathrm{Fred}_{\mathbb{R}}(X,Y)$.

If $T \in \mathrm{Fred}_{\mathbb{R}}(X,Y)$ is surjective, an orientation of $\det(T)$ is equivalent to an orientation of $\ker T$. More generally, we have that an orientation of $\det(T)$ is equivalent to an orientation of $\ker T \otimes \operatorname{coker} T$.

In future constructions, we will encounter a plethora of rank-1 graded $\mathbb{R}$-vector spaces that arise as determinants. Given a real vector space V, we define $\det(V)$ to be the top exterior power of V, which we grade by setting its degree to $\dim_{\mathbb{R}}(V)$. More generally, to any graded $\mathbb{R}$-vector space δ, we can associate a rank-1 free abelian group called the *orientation line* denoted by $|\delta|$, which is generated by the two orientations of δ modulo the relation that the sum vanishes. We will also often write $|V|$ for the orientation line associated to $\det(V)$.

Whenever we manipulate tensor products of orientation lines, we will use the Koszul sign convention: whenever we exchange the position of two line bundles σ_1 and σ_2, we must introduce the following sign of parity:

$$\sigma_1 \otimes \sigma_2 \cong (-1)^{\deg(\sigma_1)+\deg(\sigma_2)} \sigma_2 \otimes \sigma_1.$$

Note that this is the same rule that gives general graded structures their signs. For example if $f : A \to A'$ and $g : B \to B'$ are two graded homomorphisms of graded rings, then

$$(f \otimes g)(a \otimes b) = (-1)^{|g||a|} f(a) \otimes g(b)$$

since formally, we exchanged the position of g and a.

Given a $\mathbb{Z}$-graded orientation line l, we define l^{-1} to be the dual line to l with canonical map

$$l^{-1} \otimes l \to \mathbb{Z}. \tag{2.4.1}$$

Thus the lines l and l^{-1} are by definition supported in opposite degrees.

LEMMA 2.4.3 ([Abo14, Lem. 1.7.1.]). *If l is a graded orientation line, then there is a canonical isomorphism $l \cong l^{-1}$ as $\mathbb{Z}_2$ graded lines. If $l \cong l_1 \otimes l_2 \otimes l_3$, then the isomorphism induced via the composition*

$$l \cong l_1 \otimes l_2 \otimes l_3 \cong l_1^{-1} \otimes l_2^{-1} \otimes l_3^{-1} \cong l_3^{-1} \otimes l_2^{-1} \otimes l_1^{-1} \cong l^{-1}$$

differs from the canonical isomorphism by $\sum_{i<j} \deg(l_i)\deg(l_j)$.

2.4.b) Invariants for Paths of Unitary Matrices

Subsequently, we need to associate an index and a determinant line to a patch ψ_t of symplectomorphisms of $\mathbb{C}^n$ that start at the identity and end at a matrix which does not have eigenvalue 1. Using the Lie-exponential, we can rewrite ψ_t as:

$$\psi_t = \exp(A_t) \quad \text{for } A_t \in \mathfrak{sp}(2n).$$

For technical reasons, we will assume that

$$\frac{d^k A_t}{dt^k}\Big|_{t=0} = \frac{d^k A_t}{dt^k}\Big|_{t=1}. \tag{2.4.2}$$

Equip $\mathbb{C}$ with negative cylindrical polar coordinates:

$$\begin{aligned}(-\infty, +\infty) \times S^1 &\to \mathbb{C} \\ (s,t) &\mapsto e^{-s-2\pi i t}.\end{aligned}$$

Assuming (2.4.2), fix any map

$$B \in C^\infty(\mathbb{C}, \mathbb{R}^{2n\times 2n}) \tag{2.4.3}$$

with the following asymptotic behaviour at infinity:

$$B(e^{-s-2\pi i t}) = \frac{dA_t}{dt} \quad \text{if } s \ll 0.$$

Writing I for the standard complex structure on $\mathbb{C}^n$, define the operator:

$$\begin{aligned}D_\psi : W^{1,p}(\mathbb{C}, \mathbb{C}^n) &\to L^p(\mathbb{C}, \mathbb{C}^n) \\ X &\mapsto \partial_s X + I(\partial_t X - B \cdot X),\end{aligned} \tag{2.4.4}$$

where $p > 2$. Since 1 is not an eigenvalue of ψ_1, we have that D_ψ is a Fredholm operator [AD14, Sec. 8.7.].

Definition 2.4.4. *The **determinant line** of ψ is defined to be the determinant line of D_ψ; denoted $\det(D_\psi)$. We define the degree of $\det(D_\psi)$ to be equal to the Fredholm index of D_ψ*

$$ind(\psi) = \dim_{\mathbb{R}}(\ker D_\psi) - \dim_{\mathbb{R}}(\operatorname{coker} D_\psi).$$

*We call this integer the **cohomological Conley-Zehnder index** of ψ.*

Before we move on, observe that in the definition of D_ψ there was a choice involved, particularly in the one of B. We have the following important result:

Lemma 2.4.5. *Given two choices of maps B_0 and B_1, we have an isomorphism of the determinant lines:*

$$\det(D^0_\psi) \cong \det(D^1_\psi).$$

This isomorphism is canonical up to multiplication by a positive real number.

Proof. Given a path from B_0 to B_1, we obtain a path γ_0 from D^0_ψ to D^1_ψ. Pulling the determinant line bundle back over this path then leads to a line bundle over the interval with the above spaces as end points. The space of such Fredholm operators is contractible and thus given another path γ_1 we have that there exists a homotopy $\Gamma : [0,1] \times [0,1] \to \mathrm{Fred}_{\mathbb{R}}(W^{1,p}(\mathbb{C}, \mathbb{C}^n), L^p(\mathbb{C}, \mathbb{C}^n))$ such that $\Gamma(0,t) = \gamma_0(t)$ and $\Gamma(1,t) = \gamma_1(t)$. Since vector bundles over the interval are trivial, we have that for fixed $s \in [0,1]$, the path $\Gamma(s,t)$ induces an isomorphism

$$\varphi_s : \det(D^0_\psi) \to \det(D^1_\psi).$$

Two isomorphisms of a real 1-dimensional vector space differ by multiplication with a scalar and thus we have that $\varphi_0 = a_s \varphi_s$ for some $a_s \in \mathbb{R}$. Since the path $\Gamma(\cdot, t)$ is continuous for fixed t, we have that a_s depends continuously on s. In addition, a_s can never be zero since $\varphi_0 \neq 0$, thus the isomorphism is canonical up to multiplication by a positive scalar a_1. □

2.4.c) Gluing of Operators and Determinant Lines

In order to connect different orbits, we will need to discuss gluing operators on the cylinder $\mathbb{R} \times S^1$, that we will from now on denote by Z. Let $\psi_\pm$ be a pair of paths of symplectic matrices that both satisfy (2.4.2), which do not have 1 as an eigenvalue, and with associated loops of matrices $B_{\pm,t} \in \mathfrak{sp}(2n)$. Consider any matrix-valued function B on Z which agrees at the positive and negative ends with $B_{\pm,t}$, i.e. a map

$$\begin{aligned} B &: Z \to \mathfrak{gl}(2n, \mathbb{R}) \\ B_{s,t} &= B_{+,t} \quad \text{if } 0 \ll s \\ B_{s,t} &= B_{-,t} \quad \text{if } 0 \gg s. \end{aligned}$$

We have the Cauchy-Riemann operator D_B on the cylinder:

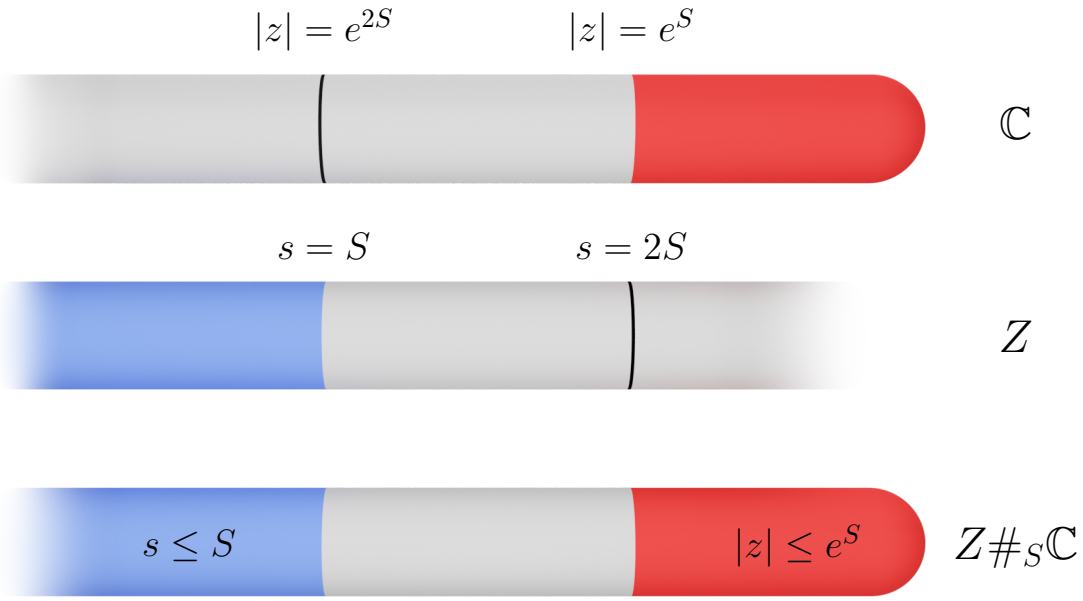
$$D_B : W^{1,p}(Z, \mathbb{C}^n) \to L^p(Z, \mathbb{C}^n) \tag{2.4.5}$$

$$X \mapsto \partial_s X + I(\partial_t X - B \cdot X). \tag{2.4.6}$$

Note that up to here, we are simply repeating the above construction for the cylinder. We now relate the determinant lines of D_B and $D_{\psi_\pm}$ through *gluing*.

Fix a real number S and consider the Riemannian surface obtained by gluing the disc $\{z \mid |z| \leq e^{2S}\}$ in $\mathbb{C}$ to the half-cylinder $(-\infty, 2S] \times S^1 \subset Z$ through the identification of the closed subsets

$$\begin{aligned} [S, 2S] \times S^1 &\to \{z \mid e^S \leq |z| \leq e^{2S}\} \\ (s, t) &\mapsto e^{3S - s - 2\pi i t}. \end{aligned}$$



We will write $Z \#_S \mathbb{C}$ for the glued space. If S is large enough, we have that the

restriction of B to $[S,\infty)\times S^1$ agrees with $B_{+,t}$. Thus we obtain an operator on $Z\#_S\mathbb{C}$ denoted by $D_B\#_S D_{\psi_+}$, which we call the *glued operator*. The properties of $D_B\#_S D_{\psi_+}$ as a Fredholm operator can be reduced to those of D_B and D_{ψ_+}. We have the following result:

LEMMA 2.4.6 ([Abo14, Lem. 1.4.5.]). *There exists an isomorphism of graded lines*

$$\det(D_B\#_S D_{\psi_+}) \cong \det(D_B)\otimes\det(D_{\psi_+})$$

which is canonical up to multiplication by a positive real number. $\square_{\text{ref.}}$

REMARK 2.4.7. From the above lemma, one can see that:

- An orientation of $\det(D_B\#_S D_{\psi_+})$ is induced by ones of $\det(D_B)$ and $\det(D_{\psi_+})$;
- The Fredholm index of the glued operator is given by the sum of the other two Fredholm indices.

Next, observe that after identifying $Z\#_S\mathbb{C}$ with $\mathbb{C}$, the operator D_{ψ_-} agrees with $D_B\#_S D_{\psi_+}$ outside of a compact set. Thus there exists a path of Fredholm operators from D_{ψ_-} to $D_B\#_S D_{\psi_+}$. Just as in the proof of Lemma 2.4.5, we see that there exists an isomorphism:

$$\det(D_{\psi_-}) \cong \det(D_B\#_S D_{\psi_+}) \tag{2.4.7}$$

that is again canonical up to multiplication by a positive real number. For what we have in mind, we will need to further weaken the assumptions on B.

DEFINITION 2.4.8. *We say that a Cauchy-Riemann operator has* ***asymptotic conditions*** *which agree with $D_{\psi_\pm}$ if the operator is given by Equation (2.4.5) and*

$$\lim_{s\to\pm\infty}|B_{s,t}-B_{\pm,t}| = 0.$$

PROPOSITION 2.4.9 ([Abo14, Prop. 1.4.6.]). *If D_B is a Cauchy-Riemann operator with asymptotic conditions $D_{\psi_\pm}$, then up to multiplication with a positive scalar, there is a canonical isomorphism*

$$D_{\psi_-} \cong \det(D_B\#_S D_{\psi_+})$$

induced by gluing.

Proof. Consider another operator B' with the same asymptotic conditions that in contrary to B agrees with $B_{\pm,t}$ for $s\ll 0$ and $s\gg 0$ respectively. Then we have that B and B' are linked by a path of operators with the same asymptotic conditions. By the usual argument, we have that $\det(B)\cong\det(B')$. Combining this with (2.4.7) concludes the proof. $\square$

2.4.d) Index of a Hamiltonian Orbit

In order to put a grading on our to be defined Floer complexes, we want to assign a homotopy class of trivialisations of $x^*(T^*M)$ to each orbit x. These trivialisations will allow us to associate a path of symplectic matrices to each loop and then consider the corresponding Conley-Zehnder index. We will make use of the following result about vector bundles:

LEMMA 2.4.10 ([Abo14, Lem. 1.4.12.]). *Let $E_{\mathbb{C}}$ be a complex vector bundle over the circle. There is a bijective correspondence between the homotopy classes of trivialisations of $E_{\mathbb{C}}$ and those of* $\det(E_{\mathbb{C}})$.

Proof. In the proof we will use the fact that every complex vector bundle over S^1 is trivial. Given two trivialisations $\psi_1, \psi_2 : S^1 \to \mathbb{C}^n$, we have that they differ by a map $\varphi : S^1 \to \mathrm{GL}(n, \mathbb{C})$, i.e. an element of the free loop space $\mathcal{L}\,\mathrm{GL}(n, \mathbb{C})$. Moreover, we have that two trivialisations are homotopic if and only if they lie in the same connected component. Since $\mathrm{GL}(n, \mathbb{C})$ is a group, its free loop space splits as a product $\mathrm{GL}(n, \mathbb{C}) \times \Omega\,\mathrm{GL}(n, \mathbb{C})$, where $\Omega\,\mathrm{GL}(n, \mathbb{C})$ is the set of loops based at the identity. Since $\pi_0(\mathrm{GL}(n, \mathbb{C})) = \{e\}$, we have that the connected component of $\mathcal{L}\,\mathrm{GL}(n, \mathbb{C})$ correspond to those of $\Omega\,\mathrm{GL}(n, \mathbb{C})$, but these are easy to identify by the isomorphism $\pi_0(\Omega\,\mathrm{GL}(n, \mathbb{C})) \cong \pi_1(\mathrm{GL}(n, \mathbb{C}))$.

Next, let $e_1, \dots, e_n$ be the global frame induced by ψ_1. By definition of φ, we have that $\varphi(e_1), \dots, \varphi(e_n)$ is the frame induced by ψ_2. We have that $\{e_1, \dots, e_n\}$ is a global frame of $\det(E)$ via $e_1 \wedge \cdots \wedge e_n$, and analogously that $\varphi(e_1) \wedge \cdots \wedge \varphi(e_n)$ provides another global frame. From linear algebra, we know that

$$\varphi(e_1) \wedge \cdots \wedge \varphi(e_n) = \det(\varphi) e_1 \wedge \cdots \wedge e_n$$

where we used the slight abuse of notation $\det(\varphi) := \det \circ \varphi$. Now comes the crux of the argument: since $\det : \mathrm{GL}(n, \mathbb{C}) \to \mathbb{C}^*$ induces an isomorphism on the level of fundamental groups, we have that two trivialisations ψ_1 and ψ_2 induce homotopic trivialisation of $\det(E)$ if and only if ψ_1 and ψ_2 are homotopic. □

Assume that $E_{\mathbb{C}}$ is the complexification of a real line bundle $E_{\mathbb{R}}$ over S^1 given by $E_{\mathbb{C}} := E_{\mathbb{R}} \otimes \mathbb{C}$. In this case, $\det(E_{\mathbb{C}})$ is the complexification of the bundle $\det(E_{\mathbb{R}})$. Any trivialisation of $E_{\mathbb{C}}$ induces a map

$$S^1 \to \mathbb{R}P^1$$

which assigns the image of the real line $\det(E_{\mathbb{R}})$ in $\mathbb{C}$ to a point in S^1. Fix the standard orientation of $\mathbb{R}P^1$ which corresponds to rotating lines in the mathematically positive sense. We call this map the *Gauss map*.

LEMMA 2.4.11 ([Abo14, Lem. 1.4.14.]). *Up to homotopy, there is a unique trivialisation of $E_{\mathbb{C}}$ such that the Gauss map has degree 0 if $E_{\mathbb{R}}$ is orientable and 1 if $E_{\mathbb{R}}$ is not orientable.*

Proof. By Lemma 2.4.10, it suffices to show the existence of a unique trivialisation of $\det(E_{\mathbb{C}})$ with these properties. In particular, we should understand how the previous map $S^1 \to \mathbb{R}P^1$ behaves under changes of trivialisation. The key point is

that the natural map $U(1) \cong S^1 \to \mathbb{R}P^1$ that sends an element $e^{it} \in \mathbb{C}^*$ to the line connecting e^{it} and 0 has degree 2. We have seen in the proof of the previous Lemma that two trivialisations of $E_{\mathbb{C}}$ differ by a map $S^1 \to \mathrm{GL}(n, \mathbb{C})$ and that this map is able to distinguish different homotopy classes of trivialisations. From the point of view of a fixed trivialisation, we have that the restriction of the bundle to the unit bundle looks like a torus sitting over S^1, and that all the other trivialisations look like 'twisted-tori'.

If we now use the degree to identify the homotopy classes of maps from S^1 to $\mathbb{R}P^1$ with $\mathbb{Z}$, we see that since changing trivialisations is done via a continuous rotation of the unit circle along S^1, it follows that the group of trivialisations acts on the homotopy classes of maps from S^1 to $\mathbb{R}P^1$ by adding an element of $2\mathbb{Z}$. In particular, assuming that $E_{\mathbb{R}}$ is orientable, we see that the degree has to be even and thus that there is a unique trivialisation for which the Gauss map has degree 0. Analogously, we have that if $E_{\mathbb{R}}$ is not orientable then there is a unique trivialisation such that the degree is equal to 1. □

By standard results, we can identify the natural symplectic structure on TT^*M with the one on $\pi^*(T^*M)_{\mathbb{C}}$, and we have:

Lemma 2.4.12 ([Abo14, Lem. 1.4.17.]). *If J_t is a family of almost complex structures on T^*M, the vector bundle $x^*(TT^*M)$ admits a trivialisation as a unitary vector bundle which is canonical up to homotopy. Furthermore, given an annulus $u : S^1 \times [0,1] \to T^*M$ there exists a trivialisation of $u^*(TT^*M)$ that is unique up to homotopy, and whose restriction to $u^*(TT^*M)|_{S^1 \times \{t\}}$ for $t \in [0,1]$ is the one provided by the first part of the lemma.*

Proof. The first part is a straight application of Lemma 2.4.11. For the second part, apply Lemma 2.4.11 to $(\pi \circ u)^*(T^*M)_{\mathbb{C}}$ and then use uniqueness up to homotopy of this very lemma. □

Assume we are given a Hamiltonian orbit x. By definition, there exists a path Φ^t of Hamiltonian symplectomorphisms such that $x(t) = \Phi^t(x(0))$ and whose differential defines a linear symplectic map:

$$D\Phi^t(x(0)) : T_{x(0)}T^*M \to T_{x(t)}T^*M.$$

On the other hand, Lemma 2.4.12 provides a trivialisation of the corresponding unitary vector bundle over the circle which identifies J_t and ω with the standard complex and symplectic structures on $\mathbb{C}^n$. In particular, we obtain a family $\psi_x(t) = \exp(A_x(t))$ of symplectomorphisms of $\mathbb{C}^n$ as the composition

$$\mathbb{C}^n \xrightarrow{\cong} T_{x(0)}T^*M \xrightarrow{D\Phi^t(x(0))} T_{x(t)}T^*M \xrightarrow{\cong} \mathbb{C}^n.$$

We can now assign an integer to each orbit:

Definition 2.4.13. *The **cohomological Conley-Zehnder index** of a non-degenerate orbit x is*

$$|x| = \mathrm{ind}(\psi_x).$$

The determinant line is the 1-dimensional $\mathbb{Z}$-graded real vector space

$$\delta_x \equiv \det(D_{\psi_x}). \tag{2.4.8}$$

The orientation line o_x is then defined to be the $\mathbb{Z}$-graded abelian group with two generators corresponding to the two orientations of δ_x, and the relation that the sum vanishes.

In Section 3.2, we will introduce a product on Floer cohomology. In order for this product to be homogeneous of degree 0, we need to introduce the following shift for the degree of certain orbits:

$$w(x) := \begin{cases} 0 & \text{if } x^*(TM) \text{ is orientable;} \\ -1 & \text{if not.} \end{cases} \tag{2.4.9}$$

Definition 2.4.14. *The **cohomological degree** of an orbit is*

$$\deg(x) = |x| - w(x).$$

2.5 Floer Cohomology of Linear Hamiltonians

Given a local system ν on the free loop space of T^*M, the goal of this section is to construct the Hamiltonian Floer cochain complex of a linear Hamiltonian H as the cohomology of a graded abelian group

$$CF^i(H;\nu) \equiv \bigoplus_{\substack{x\in\mathcal{O}(H) \\ \deg(x)=i}} o_x[w(x)] \otimes \nu_x, \tag{2.5.1}$$

where the notation $o_x[w(x)]$ indicates that we shift the degree of the graded line o_x by $w(x)$.

2.5.a) Orientations

Let x_0 and x_1 be Hamiltonian orbits and consider an element $u \in \mathcal{M}(x_0;x_1)$. By Lemma 2.4.12, we have a canonical trivialisation of $u^*(TT^*M)$. Using this trivialisation, we can identify the linearisation D_u of the Floer equation at u with an operator

$$W^{1,p}(Z,\mathbb{C}^n) \to L^p(Z,\mathbb{C}^n).$$

Since trivialisations of $u^*(TT^*M)$ restrict along the ends to the trivialisation of $x_i^*(TT^*M)$ used to define the path ψ_{x_i}, we have the following result by Proposition 2.4.9:

Theorem 2.5.1 ([Abo14, Thm. 1.5.1.]). *If $u \in \mathcal{M}(x_0;x_1)$ is a solution to Floer's equation, then there is an isomorphism of graded lines*

$$\det(D_u) \otimes \delta_{x_0} \cong \delta_{x_1}$$

which is canonical up to multiplication by a positive real number. In particular, the virtual dimension of u is

$$\deg(x_0) - \deg(x_1) - 1.$$

Let us now consider the situation where the Floer data (H, J) are regular in the sense of Equation (2.3.4).

COROLLARY 2.5.2. *There is an isomorphism*

$$|\det(D_u)| \cong |\mathbb{R} \cdot \partial_s| \otimes |\mathcal{M}(x_0; x_1)|,$$

where $|\mathcal{M}(x_0; x_1)|$ *denotes the orientation line of* $T_u\mathcal{M}(x_0; x_1)$.

Proof. Recall that $\mathcal{M}(x_0; x_1)$ is defined as the zero-level set of the Floer operator with linearisation at a solution $u \in \mathcal{M}(x_0; x_1)$ given by D_u. The implicit function theorem thus implies the following short exact sequence

$$0 \longrightarrow \mathbb{R} \cdot \partial_s u \longrightarrow \ker(D_u) \longrightarrow T_u\mathcal{M}(x_0; x_1) \longrightarrow 0.$$

This short exact sequence splits, yielding $\ker(D_u) \cong \mathbb{R} \cdot \partial_s u \oplus T_u\mathcal{M}(x_0; x_1)$ and inducing an isomorphism of line bundles

$$|\det(D_u)| = \left| \bigwedge^{\max} \ker(D_u) \right| \cong \left| \bigwedge^{\max} \mathbb{R}\cdot\partial_s \oplus \bigwedge^{\max} T_u\mathcal{M}(x_0; x_1) \right| \cong |\mathbb{R}\cdot\partial_s u| \otimes |\mathcal{M}(x_0; x_1)|.$$

□

LEMMA 2.5.3. *If* $\deg(x_0) \leq \deg(x_1)$, *and* $x_0 \neq x_1$ *then the moduli space* $\mathcal{M}(x_0; x_1)$ *is empty.*

Proof. Assume by way of contradiction that u is an element of $\mathcal{M}(x_0; x_1)$ for $x_0 \neq x_1$. Then $\partial_s u$ defines a non-zero element of the kernel of D_u and since D_u is surjective, we have that $\operatorname{ind}(D_u) > 0$. Thus

$$\begin{aligned} 0 < \operatorname{ind}(D_u) &= \deg(x_0) - \deg(x_1) - 1 \\ \Rightarrow \deg(x_0) &> \deg(x_1) \end{aligned}$$

which is a contradiction. □

Let us now assume that $\deg(x_0) = \deg(x_1) + 1$ in order to refine the count of elements in $\mathcal{M}(x_0; x_1)$ to maps on orientation lines:

LEMMA 2.5.4 ([Abo14, Lem. 1.5.4.]). *Every rigid cylinder* $u \in \mathcal{M}(x_0; x_1)$ *determines a canonical isomorphism of orientation lines*

$$\partial_u : o_{x_1} \to o_{x_0}.$$

Proof. Recall that o_{x_i} is the orientation line of δ_{x_i} of Equation (2.4.8). We have an isomorphism

$$o_{x_1} \cong |\det(D_u)| \otimes o_{x_0} \cong |\mathbb{R} \cdot \partial_s| \otimes |\mathcal{M}(x_0; x_1)| \otimes o_{x_0}.$$

After fixing the orientation of $\mathbb{R}\cdot\partial_s$ corresponding to the generator ∂_s and realising that since u is rigid the tangent space $T_u\mathcal{M}(x_0; x_1)$ has dimension 0; hence is canonically trivial, we obtain the desired isomorphism of orientation lines. □

2.5.b) Floer Cohomology

Recall that ν was a local system on the free loop space of T^*M. We can think of every holomorphic cylinder u as giving a path in the free loop space of T^*M and hence an isomorphism

$$\nu_u : \nu_{x_1} \to \nu_{x_0}.$$

Lemma 2.5.4 induces, after a shift by $w(x_1)$, a map

$$\partial_u : o_{x_1}[w(x_1)] \to o_{x_0}[w(x_0)]$$

which still has degree 1 as $w(x_0) = w(x_1)$ whenever x_0 and x_1 can be connected by a holomorphic cylinder. Keeping the definition of the Floer complex (2.5.1) we are now able to define the differential:

DEFINITION 2.5.5. *Given $x_1 \in \mathcal{O}(H)$, we define the differential:*

$$d : CF^i(H;\nu) \to CF^{i+1}(H;\nu) \tag{2.5.2}$$

$$d|_{o_{x_1}[w(x_1)]\otimes\nu_{x_1}} := \sum_{\substack{x_0\in\mathcal{O}(H)\\ \deg(x_0)=\deg(x_1)+1}} \sum_{u\in\mathcal{M}(x_0;x_1)} \partial_u \otimes \nu_u. \tag{2.5.3}$$

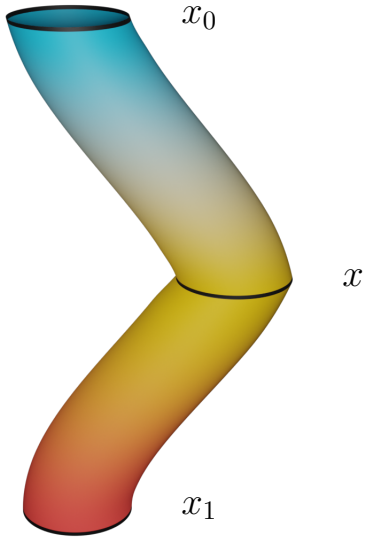
REMARK 2.5.6. If we choose a generator α for o_{x_1} and write $S^{w(x_1)}\alpha$ for the corresponding generator of $o_{x_1}[w(x_1)]$, then by definition

$$dS^{w(x_1)}\alpha := (-1)^{w(x_1)} S^{w(x_0)} d\alpha. \tag{2.5.4}$$

In order to show that this differential squares to zero, we will need the following result for closed symplectic manifolds which is still valid in our case since we used assumption on the symplectic form and almost complex structure to reduce to the closed case. We thus refer to [AD14, Thm. 9.2.1.]

THEOREM 2.5.7 ([Abo14, Thm. 1.5.6.]). *If $\deg(x_0) = \deg(x_1) + 2$, then we have that the compactified moduli space $\overline{\mathcal{M}}(x_0;x_1)$ is a 1-dimensional manifold with boundary*

$$\bigsqcup_{\deg(x)=\deg(x_1)+1} \mathcal{M}(x_0;x) \times \mathcal{M}(x;x_1).$$



Two refinements of this result will be needed:

1. If β is a homotopy class of cylinders connecting x_0 and x_1, then we have that the corresponding component $\mathcal{M}_\beta(x_0; x_1)$ of the moduli space compactifies to $\overline{\mathcal{M}}_\beta(x_0; x_1)$ whose boundary is

$$\bigsqcup_{\substack{\deg(x)=\deg(x_1)+1 \\ \beta_1 \# \beta_2 = \beta}} \mathcal{M}_{\beta_1}(x_0; x) \times \mathcal{M}_{\beta_2}(x; x_1).$$

   Here β_1 and β_2 are homotopy classes of cylinders, and $\#$ stands for the operation of gluing along the common end, which in this case consists of x. The compatibility of the Gromov compactification with the decomposition into homotopy classes follows immediately from the topology on $\overline{\mathcal{M}}(x_0; x_1)$.

2. This refinement concerns tangent spaces. Assume that a cylinder $w \in \mathcal{M}(x_0; x_1)$ lies sufficiently close to a pair of curves $(u, v) \in \mathcal{M}(x_0; x) \times \mathcal{M}(x; x_1)$. We then have a non-linear analogue of the gluing construction of Section 2.4.c) that yields a map

$$\ker(D_u) \oplus \ker(D_v) \to \ker(D_w),$$

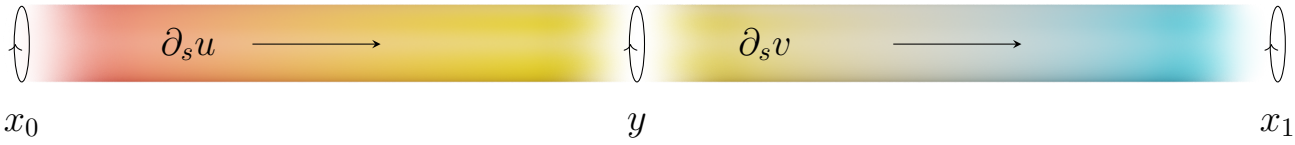
   where $\ker(D_w)$ is the middle term of the exact sequence:

$$\mathbb{R} \cdot \partial_s w \longrightarrow \ker(D_w) \longrightarrow T_w \mathcal{M}(x_0; x_1).$$

   Since by assumption w is close to the boundary stratum (u, v), it makes sense to say that an element of $T_w \mathcal{M}(x_0, x_1)$ either points towards the boundary, or away from it.

Lemma 2.5.8 ([Abo14, Lem. 1.5.7.]). *The image of $\partial_s u$ in $T_w \mathcal{M}(x_0; x_1)$ points away from the boundary, while the image of $\partial_s v$ points towards the boundary.*

$\square_{\text{ref.}}$



We are now in a position to prove the following result:

Proposition 2.5.9 ([Abo14, Prop. 1.5.10.]). *The map d of equation (2.5.2) squares to 0.*

Proof. The proof follows the template of similar results in Floer theory. We are going to show that d^2 already vanishes on each of its constituent lines, i.e. that for the map:

$$d^2|_{o_{x_2}[w(x_2)] \otimes \nu_{x_2}} = \sum_{x_1} \sum_{x_0} \sum_{\substack{u \in \mathcal{M}(x_0; x_1) \\ v \in \mathcal{M}(x_1; x_2)}} (\partial_u \circ \partial_v) \otimes (\nu_u \circ \nu_v)$$

we already have for all $x_0, x_1 \in \mathcal{O}(H)$ such that $\mathrm{ind}(x_0) - 2 = \mathrm{ind}(x_1) - 1 = \mathrm{ind}(x_0)$ that

$$\sum_{\substack{u \in \mathcal{M}(x_0;x_1) \\ v \in \mathcal{M}(x_1;x_2)}} (\partial_u \circ \partial_v) \otimes (\nu_u \circ \nu_v) = 0. \tag{2.5.5}$$

Let $[u]$ denote the homotopy class of u as a map from the cylinder to T^*M with asymptotic conditions x_0 at $s = -\infty$ and x_1 at $s = +\infty$, and similarly for $[v]$. By gluing along the end converging to x_1, we can associate to such a pair a homotopy class of cylinders with asymptotic conditions x_0 and x_2, which we denote $[u]\#[v]$.

Since ν is a local system, the composition $\nu_u \circ \nu_v$ depends only on the homotopy class $[u]\#[v]$. Thus the sum 2.5.5 further decomposes as

$$\sum_{\beta} \Big(\sum_{[u]\#[v]=\beta} (\partial_u \circ \partial_v) \otimes \nu_\beta \Big). \tag{2.5.6}$$

Thus if we can show that each summand vanishes for a fixed homotopy class of cylinders β, we have shown that $d^2 = 0$.

To show this, fix an orientation μ of $\overline{\mathcal{M}}_\beta(x_0; x_1)$, the space of trajectories from x_0 to x_2 that are homotopic to β. Similar to the proof of Lemma 2.5.4, we have the following canonical isomorphism by Theorem 2.5.1:

$$o_{x_2} \cong |\overline{\mathcal{M}}_\beta(x_0; x_2)| \otimes |\mathbb{R} \cdot \partial_s| \otimes o_{x_0}.$$

Again fixing the orientation of $\mathbb{R} \cdot \partial_s$ corresponding to the generator ∂_s, we have an induced isomorphism $o(\mu)$:

$$o_{x_2} \to o_{x_0}.$$

On the boundary of $\overline{\mathcal{M}}_\beta(x_0; x_2)$, we can compare this isomorphism with $\partial_u \circ \partial_v$. Since there are only two isomorphisms from $\mathbb{Z}$ to $\mathbb{Z}$, there are only two possibilities: either $\partial_u \circ \partial_v = o(\mu)$ or $\partial_u \circ \partial_v = -o(\mu)$. Since $\partial_u \circ \partial_v$ evaluates positively on a tangent vector to $\overline{\mathcal{M}}_\beta(x_0; x_2)$ pointing outwards at (u, v), we can identify $o(\mu)$ by evaluating it at such a tangent vector. Since $\mathrm{ind}(x_0) = \mathrm{ind}(x_2) + 2$, we have that $\overline{\mathcal{M}}_\beta(x_0; x_2)$ is a 1-dimensional oriented compact manifold. Such manifolds can be classified and they are diffeomorphic to a union of circles and closed intervals.

The broken trajectory $u\#v$ lies on the boundary of $\overline{\mathcal{M}}_\beta(x_0; x_2)$ and hence the connected component containing it is diffeomorphic to a closed interval, and thus contains a second broken trajectory with opposite orientation that we will denote by $u'\#v'$ with $u' \in \mathcal{M}(x_0; x_1)$, $v' \in \mathcal{M}(x_1; x_2)$. W.l.o.g. assume that $\partial_u \circ \partial_v = o(\mu)$, we then have that $o(\mu) = -\partial_{u'} \circ \partial_{v'}$ and thus $\partial_u \circ \partial_v = -\partial_{u'} \circ \partial_{v'}$. Note that this shows that our argument is a posteriori independent of the choice of orientation μ. This concludes the proof as it shows that the summands in (2.5.6) appear in pairs of opposite signs. □

Definition 2.5.10. *The* ***Floer cohomology*** $HF^*(H; \nu)$ *of* H *is the cohomology of* $CF^*(H; \nu)$ *with respect to the differential* d.

2.6 Symplectic Cohomology as a Limit

Let H^+ and H^- be linear Hamiltonians, which have non-degenerate orbits in the sense of Definition 2.3.13. Assuming that $H^+ \leq H^-$, where $\leq$ is the preorder of Definition 2.3.1, we shall subsequently construct a continuation map

$$\mathfrak{c} : HF^*(H^+;\nu) \to HF^*(H^-;\nu).$$

We define

$$SH^*(T^*M;\nu) \equiv \varinjlim_{\mathfrak{c}} HF^*(H;\nu) \tag{2.6.1}$$

to be the colimit of all Floer cohomology groups of linear Hamiltonians with respect to the continuation maps.

2.6.a) Energy of Pseudo-Holomorphic Maps

In the future, we will need a generalisation of the notion of energy that we previously defined to maps from an arbitrary compact Riemannian surface (Σ, j) with boundary into T^*M. We also choose a family H_z of linear Hamiltonians on T^*M, which are parametrised by $z \in \Sigma$, and a 1-form γ on Σ. The maps from Σ to T^*M that we will study will be assumed to satisfy the following holomorphic equation for every $z \in \Sigma$:

$$(Du(z) - X_{H_z} \otimes \gamma_z) \circ j = J_z \circ (Du(z) - X_{H_z} \otimes \gamma_z). \tag{2.6.2}$$

We then define the energy of a map u by:

$$E(u) = \int_\Sigma ||Du - X_{H_z} \otimes \gamma_z||^2.$$

Lemma 2.6.1. *Assume that $E(u) = 0$. The image of every tangent vector under Du is parallel to X_{H_z}. Furthermore, if the image of u intersects the complement of D^*M, we have that it is contained in a level set of ρ.*

Proof. Since $E(u)$ is the sum of the positive contributions of every point of Σ, we have that $E(u) = 0$ if and only if $||Du(z) - X_H(z) \otimes \gamma_z||^2 = 0$ for all $z \in \Sigma$. Thus fix $z \in \Sigma$ and choose a system of local holomorphic coordinates (s,t) around z. We have

$$\begin{aligned}
||Du(z) - X_H(z) \otimes \gamma_z||^2 &= \Big|\Big|\Big(\frac{\partial u}{\partial t}\Big|_z - X_H(z) \cdot \gamma_z\Big(\frac{\partial}{\partial t}\Big|_z\Big)\Big) \otimes dt_z \\
&\quad + \Big(\frac{\partial u}{\partial s}\Big|_z - X_H(z) \cdot \gamma_z\Big(\frac{\partial}{\partial s}\Big|_z\Big)\Big) \otimes ds_z\Big|\Big|^2 \\
&= \Big|\Big|\frac{\partial u}{\partial t}\Big|_z - X_H(z)\gamma_z\Big(\frac{\partial}{\partial t}\Big|_z\Big)\Big|\Big|_{J_t}^2 + \Big|\Big|\frac{\partial u}{\partial s}\Big|_z - X_H(z)\gamma_z\Big(\frac{\partial}{\partial s}\Big|_z\Big)\Big|\Big|_{J_t}^2.
\end{aligned}$$

Thus if $E(u) = 0$, we have for $a, b \in \mathbb{R}$:

$$Du(z)\Big[a\frac{\partial}{\partial t}\Big|_z + b\frac{\partial}{\partial s}\Big|_z\Big] = a\frac{\partial u}{\partial t}\Big|_z + b\frac{\partial u}{\partial s}\Big|_z = \Big(a\gamma_z\Big(\frac{\partial u}{\partial t}\Big|_z\Big) + b\gamma_z\Big(\frac{\partial u}{\partial s}\Big|_z\Big)\Big) \cdot X_H(z),$$

which shows that D_u is parallel to X_H.

For the second part, we have for $z \in \Sigma$ with $u(z) \in (D^*M)^c$ that:

$$\begin{aligned}\frac{\partial}{\partial t}\Big|_z (\rho \circ u) = Du(z)\Big[\frac{\partial u}{\partial t}\Big|_z\Big](\rho) &= \frac{\partial u}{\partial t}\Big|_z (\rho) \\ &= d\rho_{u(z)}\Big(\frac{\partial u}{\partial t}\Big|_z\Big) \\ &= \omega_{u(z)}\Big(\frac{\partial u}{\partial t}\Big|_z, X_\rho(u(z))\Big) \\ &= \omega_{u(z)}\Big(\gamma_z\Big(\frac{\partial}{\partial t}\Big|_z\Big) \cdot X_\rho(u(z)), X_\rho(u(z))\Big) \\ &= 0.\end{aligned}$$

And analogously for $\frac{\partial}{\partial s}\Big|_z (\rho \circ u)$. □

We say that $d\gamma$ is non-positive if

$$d\gamma(\xi, j\xi) \leq 0 \tag{2.6.3}$$

for every tangent vector along Σ. In this setting a similar result to the one for the geometric energy holds:

Lemma 2.6.2. *If $d\gamma$ is non-positive, then*

$$0 \leq E(u) \leq \int_{\partial\Sigma} u^*(\lambda) - (H \circ u)\gamma.$$

Moreover, the second inequality is strict unless $d\gamma \equiv 0$.

Proof. As before, we reduce the proof to the local case by choosing local coordinates around a point. However, this time we have to make sure that the induced complex structure coincides with the one $\mathbb{R}^2$. Using equation (2.6.2), we reduce to

$$E(u) = \int_\Sigma u^*(\omega) - u^*(dH) \wedge \gamma$$

just like in the calculations above Lemma 2.3.5. But this is then equal to

$$\begin{aligned}E(u) &= \int_\Sigma d(u^*(\lambda)) - d(u^*(H) \cdot \gamma) + u^*(H) \cdot d\gamma \\ &= \int_{\partial\Sigma} u^*(\lambda) - H \circ u \cdot \gamma + \int_\Sigma u^*(dH) \cdot d\gamma\end{aligned}$$

The last term is equal to 0 if $d\gamma = 0$ and strictly less than zero if $d\gamma \neq 0$ because of the non-positivity. □

2.6.b) Continuation Maps

In this part, we will construct a continuation map

$$\mathfrak{c} : HF^*(H^+;\nu) \to HF^*(H^-;\nu)$$

assuming that $H^+ \leq H^-$. For this, a plethora of choices must be made. Firstly, we choose a family of monotonically decreasing slopes b_s, which agree with the slope of H^- whenever $s \ll 0$ and with that of H^+ if $s \gg 0$. Secondly, we choose a family $H_{s,t}$ of time-dependent Hamiltonians such that

$$\begin{aligned} H_{s,t} &= b_s\rho \ \text{ in a neighbourhood of } \rho^{-1}([1,+\infty)), \\ H_{s,t} &= H_t^{\pm} \text{ whenever } s \ll 0 \text{ resp. } s \gg 0. \end{aligned}$$

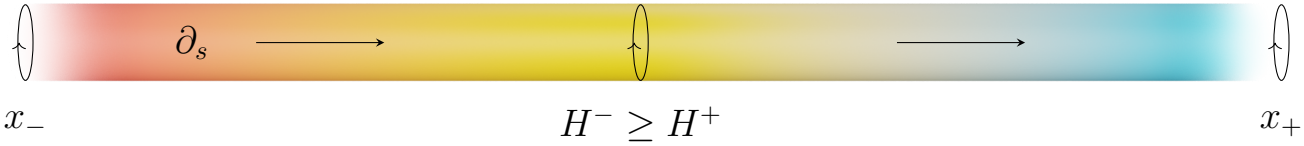
Finally, we choose a family of almost complex structures $J_{s,t}$ that satisfy the following conditions:

$$J_{s,t} \text{ is convex near } S^*M \text{ and agrees with } J_t^{\pm} \text{ whenever } s \ll 0 \text{ or } s \gg 0. \quad (2.6.4)$$

In the usual coordinates on the cylinder, the continuation equation takes on the following form:

$$J_{s,t}\frac{\partial u}{\partial s} = \frac{\partial u}{\partial t} - X_{s,t}(u(s,t)). \quad (2.6.5)$$

Note that this equation interpolates between Floer's equation for the Hamiltonians H^+ and H^-. In particular, the natural asymptotic conditions are time-1 orbits of $X_{H^\pm}$ near $s = \pm\infty$.



Definition 2.6.3. *Given a pair of orbits $x_\pm \in \mathcal{O}(H^\pm)$, the **continuation moduli space** $\mathcal{K}(x_-;x_+)$ is the space of solutions to (2.6.5) which exponentially converge to $x_\pm$ at $s = \pm\infty$.*

The moduli space $\mathcal{K}(x_-;x_+)$ presents a lot of similarities to $\mathcal{M}(x_0;x_1)$ with the major difference that there is a priori no reason for there to be an $\mathbb{R}$-action that acts by translation. The space $\mathcal{K}(x_-;x_+)$ can be described as the solution set of a Fredholm section on a Banach space. By linearisation, we can associate to each solution u an operator

$$D_u : W^{1,p}(Z, u^*(T^*M)) \to L^p(Z, u^*(T^*M))$$

whose kernel and cokernel control the structure of the moduli space near u. Analogously to Theorem 2.5.1, we have the following result:

LEMMA 2.6.4. *Given an element $u \in \mathcal{K}(x_-; x_+)$, there is a canonical isomorphism*

$$\det(D_u) \otimes \delta_{x_+} \to \delta_{x_-}. \tag{2.6.6}$$

As the above map is an isomorphism of graded lines, in addition to $\det(D_u)$ being supported in degree $\mathrm{ind}(D_u)$ and $\delta_{x_\pm}$ in degree $\deg(x_\pm)$, we can deduce that

$$\mathrm{ind}(D_u) = \deg(x_+) - \deg(x_-).$$

The following genericity result holds:

LEMMA 2.6.5 ([Abo14, Lem. 1.6.6.]). *For generic data $J_{s,t}$ and $H_{s,t}$, all moduli spaces $\mathcal{K}(x_-; x_+)$ are regular.*

Under this condition, the moduli space $\mathcal{K}(x_-; x_+)$ becomes a smooth manifold of dimension $\deg(x_+) - \deg(x_-)$. If $\deg(x_+) = \deg(x_-)$, $\mathcal{K}(x_-; x_+)$ is a 0-dimensional manifold and we call such solutions *rigid curves*.

The analogue of Lemma 2.5.4 holds, and its proof is easier since there is no quotient procedure in the definition of $\mathcal{K}(x_-; x_+)$.

LEMMA 2.6.6 ([Abo14, Lem. 1.6.8.]). *Every rigid solution to the continuation equation $u \in \mathcal{K}(x_-; x_+)$ determines a canonical isomorphism of orientation lines*

$$\mathfrak{c}_u : o_{x_+}[w(x_+)] \to o_{x_-}[w(x_-)].$$

□

In order to apply the Gromov-Floer construction, we will once again reduce to the compact case.

LEMMA 2.6.7. *Under the above assumptions, we have that all elements of $\mathcal{K}(x_-; x_+)$ have image lying in the unit disc.*

Proof (Sketch). We are going to use Lemma 2.6.2 with $\gamma := b_s dt$. Since the slopes of b_s were chosen to be decreasing, we have that $d\gamma$ is a non-positive 2-form on Z. Thus

$$0 < \int_{\partial\Sigma} u^*(\lambda) - H \circ u \cdot b_s dt.$$

The proof then is essentially word by word the same as the proof of Lemma 2.3.7, where we highlight that one needs to use that $J_{s,t}$ is convex near S^*M. □

Due to this result, we are able to define the map:

$$\mathfrak{c} : CF^*(H^+; \nu) \to CF^*(H^-; \nu)$$

through

$$\mathfrak{c} := \sum \mathfrak{c}_u \otimes \nu_u.$$

This sum is finite by Gromov compactness, whose proof relies on the fact that solutions to the continuation equation stay in a compact set.

If $\deg(x_+) = \deg(x_-) + 1$, the compactification $\overline{\mathcal{K}}(x_-; x_+)$ obtained by the Gromov-Floer compactification presents a decomposition of its boundary into two types of boundary strata

$$\bigsqcup_{x_0:\deg(x_-)=\deg(x_0)} \mathcal{K}(x_-; x_0) \times \mathcal{M}(x_0; x_+),$$
$$\bigsqcup_{x_1:\deg(x_+)=\deg(x_1)} \mathcal{M}(x_-; x_1) \times \mathcal{K}(x_1; x_+).$$

This is for instance the content of [AD14, Thm. 11.1.10.]. Note that the first disjoint union corresponds exactly to the elements involved in $\mathfrak{c} \circ d$ that connect x_- and x_+, and the second disjoint union to that of $d \circ \mathfrak{c}$.

Proposition 2.6.8 ([Abo14, Prop. 1.6.10.]). *The map $\mathfrak{c}$ is a chain homotopy*

$$\mathfrak{c} \circ d - d \circ \mathfrak{c} = 0.$$

Proof (Sketch). The proof presented follows the proof of 2.5.9 almost verbatim and we will thus focus on the hardest part; showing that both terms indeed have opposite signs.

Let H^- and H^+ be two Hamiltonians such that continuation maps make sense. Let $x_1 \in \mathcal{O}(H^-)$ and $y_1, y_2 \in \mathcal{O}(H^+)$ such that $\deg(x_1) = \deg(x_2)$ and $\deg(x_1) = \deg(y_1) - 1$. Elements of the form $u_1 \in \mathcal{K}(x_1, y_1)$ and $v_+ \in \mathcal{M}(y_1, y_2)$ induce isomorphisms

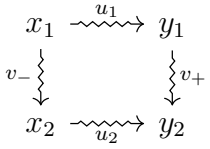
$$\det(D_{u_1}) \otimes \delta_{y_1} \cong \delta_{x_1}$$
$$\det(D_{v_+}) \otimes \delta_{y_2} \cong \delta_{y_1}$$

that when taking all possible combinations yield the map $\mathfrak{c} \circ d$. Composing the two isomorphisms yields

$$\det(D_{u_1}) \otimes \det(D_{v_+}) \otimes \delta_{y_2} \cong \delta_{x_1}.$$

Since $\deg(x_1) = \deg(y_1)$, we have that $\det(D_{u_1})$ is canonically trivial. On the other hand, the construction of the differential depended on a trivialisation corresponding to $\partial_s v_+$. Similar to Lemma 2.5.8, $\partial_s v$ gives rise to a tangent vector to $\overline{\mathcal{K}}(x_1, y_2)$ that points towards the boundary.

Because of the structure of the boundary of the 1-dimensional manifold with boundary $\overline{\mathcal{K}}(x_1; y_2)$, we must have that for every pair $u_1 \in \mathcal{K}(x_1; y_1)$ and $v_+ \in \mathcal{M}(y_1; y_2)$, there must exist an orbit $x_2 \in \mathcal{O}(H^-)$ such that $\deg(x_2) = \deg(y_2)$, together with solutions $u_2 \in \mathcal{K}(x_2; y_2)$ and $v_- \in \mathcal{M}(x_1, x_2)$ such that the boundary components that (u_1, v_+) and (v_-, u_2) correspond to have opposite orientation. We can illustrate this pictorially by



where this diagram is by no means commutative.[2] Analogously to the above, a pair (v_-, u_2) induces an isomorphism

$$\det(D_{v_-}) \otimes \det(D_{u_2}) \otimes \delta_{y_2} \cong \delta_{x_1}.$$

The vector field $\partial_s u$ now gives rise to a tangent vector to $\overline{\mathcal{K}}(x_1, y_2)$ which points away from the boundary, and thus

$$\mathfrak{c} \circ d + (-d \circ \mathfrak{c}) = 0.$$

□

We have the following important result:

LEMMA 2.6.9 ([Abo14, Lem. 1.6.13.]). *The continuation map*

$$\mathfrak{c} : HF^*(H^+; \nu) \to HF^*(H^-; \nu)$$

does not depend on the choice of family $(H_{s,t}, J_{s,t})$.

Proof (Sketch). We will show this lemma by reducing it to the case of $\mathbb{Z}_2$-coefficients as for example presented in [Sal97]. Let $(H^0_{s,t}, J^0_{s,t})$ and $(H^1_{s,t}, J^1_{s,t})$ be two choices of continuation data. Individually, these families respectively define the continuation maps

$$\mathfrak{c}_i : CF^*(H^+; \nu) \to CF^*(H^-; \nu)$$

for $i \in \{0, 1\}$. Consider an interpolating family of Floer data $(H^r_{s,t}, J^r_{s,t})$ for $r \in [0, 1]$ which agrees with (H^+, J^+_t) in the limit $s \gg 0$, and with (H^-_t, J^-_t) in the limit $s \ll 0$, independent of r. Next, we claim that these families of Floer data induce a chain homotopy $\mathfrak{h}$ between $\mathfrak{c}_0$ and $\mathfrak{c}_1$. We will use a cobordism-type argument to show the claim. Let $\mathcal{K}_r(x_-, x_+)$ denote the moduli space of continuation maps for the Floer data $(H^r_{s,t}, J^r_{s,t})$, and set

$$\widetilde{\mathcal{K}}(x_-; x_+) := \bigsqcup_{r \in [0,1]} \mathcal{K}_r(x_-; x_+).$$

One can then topologise this space through its embedding in $[0, 1] \times C^\infty(Z, T^*M)$ as the intersection of the zero section and a section of a bundle whose fibre at (r, u) is simply $C^\infty(Z, u^*(TT^*M))$. Note that the fibre is independent of the parameter r, and only depends on u. In contrast, the actual section however greatly depends on r; the restriction to $\{r\} \times C^\infty(Z, T^*M)$ is given by the continuation map operator

$$u \mapsto \frac{\partial u}{\partial s} + J^r_{s,t}\Big(\frac{\partial u}{\partial t} - X^r_{s,t}(u(s,t))\Big), \tag{2.6.7}$$

where $X^r_{s,t}$ is the Hamiltonian flow associated to $H^r_{s,t}$.

By standard results, we have that for a generic choice of families $(H^r_{s,t}, J^r_{s,t})$, the moduli space $\widetilde{\mathcal{K}}(x_-, x_+)$ is regular, and hence a smooth manifold with boundary

$$\partial\widetilde{\mathcal{K}}(x_-, x_+) = \mathcal{K}_0(x_-, x_+) \cup \mathcal{K}_1(x_-, x_+)$$

[2] For once.

of dimension $\deg(x_+) - \deg(x_-) + 1$. In this setting, regularity of the moduli space at a point (r, u) is equivalent to the surjectivity of the Fredholm map

$$T_r[0,1] \oplus W^{1,p}(Z, u^*(TT^*M)) \to L^p(Z, u^*(TT^*M)),$$

where the second summand is the linearised operator D_u, and the first summand is obtained by taking the derivative of (2.6.7) with respect to the r-variable:

$$\partial_r \mapsto \frac{\partial J^r_{s,t}}{\partial r}\Big(\frac{\partial u}{\partial t} - X^r_{s,t}(u(s,t))\Big) + J^r_{s,t}\frac{\partial X^r_{s,t}}{\partial r}.$$

Regularity of the pair (r, u) then implies the existence of a canonical isomorphism

$$|T_u\widetilde{\mathcal{K}}(x_-; x_+)| \cong |T_r[0,1]| \otimes |\det(D_u)|$$

of orientation lines. Fixing an orientation of $[0,1]$ and using the isomorphism of Equation (2.6.6), this yields

$$|T_u\widetilde{\mathcal{K}}(x_-; x_+)| \otimes o_{x_+} \cong o_{x_-}.$$

When $\deg(x_+) = \deg(x_-) - 1$, we have that $\widetilde{\mathcal{K}}(x_-; x_+)$ is 0-dimensional, and thus there is an isomorphism

$$\mathfrak{h}_u : o_{x_+}[w(x_+)] \to o_{x_-}[w(x_-)].$$

In analogy to the constructions before, we then define the map $\mathfrak{h}$ to be the sum of $\mathfrak{h}_u \otimes \nu_u$ for all possible u, where $\nu_u : \nu_{x_+} \to \nu_{x_-}$ is the map induced by parallel transport.

The proof that $\mathfrak{h}$ is a chain map then follows by analysing the boundary components of the compactification of $\widetilde{\mathcal{K}}(x_-; x_+)$ in the case where $\dim \widetilde{\mathcal{K}}(x_-; x_+) = 1$, i.e. when $\deg(x_+) = \deg(x_-)$. Having constructed the chain homotopy $\mathfrak{h}$, one can then readily lift the argument for the $\mathbb{Z}_2$-case provided for example in [Sal97]. □

In addition, the continuation maps are functorial:

Proposition 2.6.10 ([Abo14, Prop. 1.6.15.]). *Given a triple of linear Hamiltonians such that $H^+ \leq H^0 \leq H^-$, we have a commutative diagram*

$$\begin{array}{ccc} HF^*(H^-;\nu) & \longrightarrow & HF^*(H^0;\nu) \\ & \searrow & \downarrow \\ & & HF^*(H^+;\nu) \end{array}$$

for any choice of continuation map.

Proof (Sketch). Let $H^+_{s,t}$ and $H^-_{s,t}$ be families of Hamiltonians which define continuous maps

$$\mathfrak{c}_{+\to 0} : HF^*(H^+;\nu) \to HF^*(H^0;\nu),$$
$$\mathfrak{c}_{0\to -} : HF^*(H^0;\nu) \to HF^*(H^-;\nu).$$

By construction, whenever $s \ll 0$, we have a that $H^+_{s,t} = H^0_t$ and if $s \gg 0$, we have that $H^-_{s,t} = H^0_t$. Given a large negative number S, we can define a new family of Hamiltonians obtained by concatenation:

$$H^S_{s,t} := \begin{cases} H^+_{s-2S,t} & \text{if } s \leq S, \\ H^-_{s,t} & \text{if } S \leq s. \end{cases}$$

Note that each such choice of S gives rise to a continuous map

$$\mathfrak{c}^S_{+\to-} : HF^*(H^+;\nu) \to HF^*(H^-;\nu).$$

In order to compare this map to the composition $\mathfrak{c}_{+\to 0} \circ \mathfrak{c}_{0\to-}$, we consider a triple of orbits x_+, x_0 and x_- of the same degree of the Hamiltonians H^+, H^- and H^- respectively. We also consider the space of pairs of solutions to the continuation map equation:

$$\mathcal{K}(x_-;x_0) \times \mathcal{K}(x_0;x_+).$$

For $S \ll 0$, we can glue pairs of such solutions and obtain a solution to the continuation map equation of $H^S_{s,t}$. Denote by $\mathcal{K}_S(x_-;x_+)$ the space of such solutions. A compactness argument similar to the one of [Sal97, Lem. 3.11] then shows that there are actually no other solutions other than glued ones, and thus there exists a bijection

$$\mathcal{K}_S(x_-;x_+) \cong \bigsqcup_{x_0} \mathcal{K}(x_-;x_0) \times \mathcal{K}(x_0;x_+).$$

The product of moduli spaces on the right hand side defines the composition of continuation maps. Thus

$$\mathfrak{c}^S_{+\to-} = \mathfrak{c}_{+\to 0} \circ \mathfrak{c}_{0\to-}.$$

We conclude the proof by recalling that the continuation maps do not depend on the family of Hamiltonians chosen by Lemma 2.6.9. □

The next lemma investigates what happens when $(H_{s,t}, J_{s,t})$ do not depend on s.

LEMMA 2.6.11. *Assume that the Floer data (H_t, J_t) is regular in the sense of Theorem 2.3.16. Solutions to the continuation map given by the data $H_{s,t} = H_t$ and $J_{s,t} = J_t$ are all regular and the only rigid solutions are independent of s. Furthermore, the corresponding continuation map is the identity on the cochain level, and hence also on cohomology.*

Proof. If $H_{s,t} = H_t$ and $J_{s,t} = J_t$, then the continuation equation (2.6.5) transforms into

$$J_t \partial_s u = \partial_t u - X_t(u(s,t)),$$

which is the Floer equation, thus the regularity part follows immediately since (H_t, J_t) is regular. If u is a solution to Equation (2.6.5), it must have endpoints $x_0, x_1 \in \mathcal{O}(H)$. Since u is assumed to be a rigid curve, we have that $\deg(x_0) = \deg(x_1)$ but this implies that $x_0 = x_1$ by Corollary 2.2.2. Thus $u(s,t) = x_0(t)$, and in particular is u independent of s. This implies that the trivialisations of $x_0^*(TT^*M)$ obtained by restricting both sides of $u^*(TT^*M)$ are indeed the same and thus the isomorphism of Lemma 2.6.6 is in fact the identity. Since $\mathcal{M}(x_0;x_1) = \varnothing$

if $\deg(x_0) = \deg(x_1)$ and $x_0 = x_1$, this is indeed the only non-zero component in the sum that defines $\mathfrak{c}$. Thus we are able to conclude that the continuation map is the identity in this case. □

Corollary 2.6.12 ([Abo14, Cor. 1.6.16.]). *If H and H' have the same slope, then there is a canonical isomorphism*

$$HF^*(H;\nu) \to HF^*(H';\nu)$$

induced by the continuation maps.

Proof. Consider the diagram

$$\begin{array}{ccc} HF^*(H;\nu) & \longrightarrow & HF^*(H';\nu) \\ & \searrow^{\mathrm{id}} & \downarrow \\ & & HF^*(H;\nu), \end{array}$$

where the identity is a valid continuation map by Lemma 2.6.11. This shows that the first map is injective, and the second map is surjective. Reversing the roles of H and H' proves that the map is an isomorphism.

In order to show that the map is canonical, consider

$$\begin{array}{ccc} HF^*(H;\nu) & \longrightarrow & HF^*(H';\nu) \\ & \searrow & \downarrow \mathrm{id} \\ & & HF^*(H';\nu). \end{array}$$

Commutativity of this diagram for every continuation map shows the claim. □

Lastly, we want to give a slightly more concrete way to compute the symplectic cohomology defined in Equation (2.6.1):

Lemma 2.6.13 ([Abo14, Lem. 1.6.17.]). *If H^i is any sequence of Hamiltonians on T^*M whose slope is unbounded, the natural map*

$$\varinjlim_i HF^*(H^i;\nu) \to SH^*(T^*M;\nu)$$

is an isomorphism.

Proof. We begin by shedding some more light on the colimit construction of Equation (2.6.1). The colimit is equal to the quotient of

$$\bigoplus_H HF^*(H;\nu)$$

by the subspace generated by $a - \mathfrak{c}(b)$ for each element $a \in HF^*(H;\nu)$ and $b \in HF^*(H';\nu)$ for a Hamiltonian H' with $H' \leq H$. The surjectivity of the natural map then immediately follows from the fact that for every H there exists a Hamiltonian H^i such that $H \leq H^i$ and thus a continuation map

$$HF^*(H;\nu) \to HF^*(H^i;\nu).$$

More precisely, we have that every element of $HF^*(H;\nu)$ is equivalent to an element of $HF^*(H^i;\nu)$ in the colimit, since every element of $HF^*(H;\nu)$ has an image in $HF^*(H^i;\nu)$.

In order to prove injectivity, let us temporarily denote the natural map by

$$f : \varinjlim_i HF^*(H^i;\nu) \to SH^*(T^*M;\nu),$$

and a continuation map between two Hamiltonians H and K by $\mathfrak{c}_{H,K}$. Let $[a],[b] \in \varinjlim_i HF^*(H^i;\nu)$ which are represented by elements $a \in HF^*(H^i;\nu)$ and $b \in HF^*(H^j;\nu)$. Assume that $f(a) = f(b)$. Thus there exists a Hamiltonian K such that $H^i \leq K$ and $H^j \leq K$ and $\mathfrak{c}_{H^i,K}(a) = \mathfrak{c}_{H^j,K}(b)$. Since the sequence $\{H^i\}$ has unbounded slope, we can find a Hamiltonian H^k with $K \leq H^k$. By Lemma 2.6.11 we have the commutative diagram

$$\begin{array}{ccc} HF^*(H^i;\nu) & \longrightarrow & HF^*(K;\nu) \\ & \searrow & \downarrow \\ & & HF^*(H^k;\nu) \end{array}$$

and so $\mathfrak{c}_{H^i,H^k}(a) = \mathfrak{c}_{H^j,H^k}(b)$, implying that $[a] = [b]$ in $\varinjlim_i HF^*(H^i;\nu)$. □

3 Operations in Symplectic Cohomology

3.1 Introduction

Symplectic cohomology can be equipped with a number of operations that stem from counts of Riemannian surfaces with at least one negative end and an arbitrary number of positive ends:

- Associated to surfaces with one negative end and no positive end, i.e. planes, we will define an element $e \in SH^0(T^*M;\mathbb{Z})$ that will serve as a unit in a to be defined ring structure.
- To a pair of pants (surfaces with one negative end and two positive ends), we will associate a product

$$* : SH^*(T^*M;\mathbb{Z}) \otimes SH^*(T^*M;\mathbb{Z}) \to SH^*(T^*M;\mathbb{Z}).$$

In his monograph, Abouzaid shows that symplectic cohomology can be equipped with the structure of a Batalin-Vilkovisky algebra (BV algebra for short). A (twisted) BV structure can be summarised by the existence of an element

$$e \in SH^0(T^*M;\mathbb{Z}),$$

a map

$$\Delta : SH(T^*M;\mathbb{Z}) \to SH^{*-1}(T^*M;\mathbb{Z})$$

called the BV operator, together with a product

$$* : SH^*(T^*M;\mathbb{Z}) \otimes SH^*(T^*M;\mathbb{Z}) \to SH^*(T^*M;\mathbb{Z}).$$

In addition, these structures need to satisfy the following compatibility relations:

1. $\Delta^2 = 0$,
2. $\Delta(e) = 0$,
3. $(a * b) * c = a * (b * c)$,
4. $a * b = (-1)^{\deg(a)\deg(b)+w(a)w(b)} b * a$,
5. $e * a = a * e = a$,

and lastly, the (twisted) BV equation:

6. $\Delta(a*b*c) + \Delta(a)*b*c + (-1)^{\deg(a)} a*\Delta(b)*c + (-1)^{\deg(a)+deg(b)} a*b*\Delta(c) = \Delta(a*b)*c + (-1)^{\deg(a)} a*\Delta(b*c) + (-1)^{(1+\deg(a))\deg(b)+w(a)w(b)} b*\Delta(a*c)$.

Remark 3.1.1. The introduction of the shift w in Equation 2.4.9 now leads to the unfortunate consequence that we are not quite presented with the structure of a BV algebra and we have to take the extra signs introduced by the shift into account. We thus end up with something that Abouzaid calls a twisted BV algebra. We will however mostly ignore this and still call it a BV structure.

In light of what is to follow in the second part of this thesis, we will focus our efforts on solely proving the existence of a ring structure [3]. We thus refer to [Abo14, Ch. 2] for the proofs that involve the BV operator Δ.

[3] With unit.

The System of Local Coefficients

As our final isomorphism will involve twisted coefficients, we need to introduce a special local system that induces a BV structure on symplectic cohomology. The local system will originate from *transgressing* a vector bundle E on the base manifold M. If M is orientable, a suitable system of local coefficients can be provided by the local system on $\mathcal{L}M$ whose fibre at a loop x is given by the set of trivialisations of $x^*(E)$.

As presented in [KT89], this set is in bijection with the set of Spin structures on $x^*(E)$. Since we are however not restricting ourselves to orientable manifolds, we need to extend this definition. It is a fact that the sum of a non-orientable bundle and its determinant bundle becomes orientable. To illustrate this, consider the Möbius bundle over the circle. Adding another dimension to it by adding a line bundle enables one to reduce the structure group from $O(2)$ to $SO(2)$; reflection in 1-dimension becomes a rotation in 2-dimensions. Analogously, the same construction works by adding an odd number of line bundles and thus the bundle $E \oplus 3\det(E)$ is orientable.

As mentioned above, the set of trivialisations of $x^*(E \oplus 3\det(E))$ is equivalent to a Spin structure on $x^*(E \oplus 3\det(E))$. This can then be reformulated in terms of our original bundle by observing that a Spin structure on $x^*(E \oplus 3\det(E))$ is equivalent to a Pin^+ structure on $x^*(E)$, as shown in [KT89, Lem. 1.7]. We are now able to define the local system:

Definition 3.1.2. *Given a real vector bundle E on M, define σ^E to be the local system on $\mathcal{L}M$ whose fibre at a loop x is the rank-1 free abelian group generated by the set of trivialisations of $x^*(E \oplus 3\det(E))$ up to homotopy.*

Note that since $E \oplus 3\det(E)$ is orientable, we have that any two trivialisations compatible with this orientation differ by the action of the set of maps from S^1 to $SO(n+3)$. As $\pi_1(SO(n+3)) = \mathbb{Z}_2$, there are exactly two such trivialisations up to homotopy.

Associated to the local system σ^E, there exist two structure maps that will be of relevance:

- A retraction of x to a point induces a canonical map
$$\mathbb{Z} \to \sigma^E_x \tag{3.1.1}$$
since such a retraction determines an extension of $x^*(E)$ to the disc, and any vector bundle over the disc is trivial.

- A map u from a pair of pants to T^*M with asymptotic conditions x_1, x_2 and x_0 at the three punctures, we can associate a map
$$\sigma^E_u : \sigma^E_{x_1} \otimes \sigma^E_{x_2} \to \sigma^E_{x_0} \tag{3.1.2}$$
that is induced by the fact that a pair of pants deformation retracts onto the figure eight. Thus a trivialisation of a vector bundle over a pair of pants is determined by its restriction to two of the three circles at infinity.

The results of this chapter can be cast in the following theorem:

THEOREM 3.1.3. *Symplectic cohomology with coefficients in σ^E is naturally equipped with the structure of a twisted, graded-commutative ring with unit.*

The existence of a product for Floer homology was first observed by Donaldson immediately after the inception of these groups. In [Vit99], Viterbo noticed that this product descends to symplectic cohomology. We will follow [Abo14, Sec.2] closely for the following account.

3.2 The Pair of Pants Product

In order to define a product structure on symplectic cohomology, we need an object that is able to connect three Hamiltonian orbits. Such a structure is for example given by a pair of pants: Let z_0, z_1 and z_2 be three distinct points on a Riemannian sphere $\mathbb{C}P^1$. We will denote by P the open Riemannian surface obtained by removing these three points from $\mathbb{C}P^1$. In order to specify what we mean by linking Hamiltonian orbits, we introduce the notion of *positive half cylinders*:

$$Z^+ := [0, +\infty) \times S^1,$$

a Riemannian surface with boundary and coordinates (s, t) with complex structure $j\partial_s = \partial_t$.

In his dissertation [Sch95], Schwarz was able to generalise the analytical concept of Floer homology from the cylinder to arbitrary Riemannian surfaces which are either closed or have ends which are endowed with the standard cylindrical structure.

DEFINITION 3.2.1. *A **positive cylindrical end** near z_i is a holomorphic embedding*

$$\varepsilon_i : Z^+ \to P$$

*which extends to a map from the disc to $\mathbb{C}P^1$ taking the origin to z_i. A **negative cylindrical end** is one whose source is the negative half cylinder $Z^- := (-\infty, 0] \times S^1$.*

At this point we make the definite choice that z_0 shall be a negative end, and z_1 and z_2 two positive ends.

The goal of any potential equation defined on P should be that its finite energy solutions converge to Hamiltonian orbits at the cylindrical ends. One way of ensuring this is to impose that the pullback of this equation under ε_i agrees with Floer's equation (2.0.1) whenever $|s|$ is sufficiently close to infinity. In order to define Floer's equation on the respective cylindrical ends, we need the following data:

1. a 1-form on Z^+ or Z^-, which we will fix to be dt,
2. a linear Hamiltonian H^i of slope b_i for each i,

3. an S^1- family of almost complex structures J^i which are convex near S^*M.

In addition, we impose the following technical assumptions on the structures on P:

4. a 1-form α whose pullback under ε_i is dt for all i,
5. a family H^P of linear Hamiltonians, parametrised by points of P such that $H^P_{\varepsilon_i(s,t)} = H^i_t$ for all i,
6. a family J^P of almost complex structures which are convex near S^*M, parametrised by points of P such that $J^P_{\varepsilon_i(s,t)} = J^i_t$.

All of these assumptions will allow us to reduce the upcoming proofs to ones already encountered in Chapter 2.

Recall that while proving compactness for the moduli spaces of cylinders, we assumed *non-positivity* of $d\gamma$ (see Equation (2.6.3)). Writing $b^P : P \to \mathbb{R}$ for the function whose value at z is the slope of H^P_z, we will require in complete analogy to the non-positivity that

$$d(b^P \cdot \alpha) \leq 0. \tag{3.2.1}$$

This will ensure compactness of the moduli space of pairs of pants. We are now in a position to formulate the Cauchy-Riemann type equation on P.

Definition 3.2.2. *The* ***moduli space of holomorphic pairs of pants*** $\mathcal{P}(x_0; x_1, x_2)$ *with input a pair of Hamiltonian orbits x_1 and x_2 of H^1 and H^2 and output a Hamiltonian orbit x_0 of H^0 is the space of maps $u : P \to T^*M$ with finite energy, i.e. maps u such that*

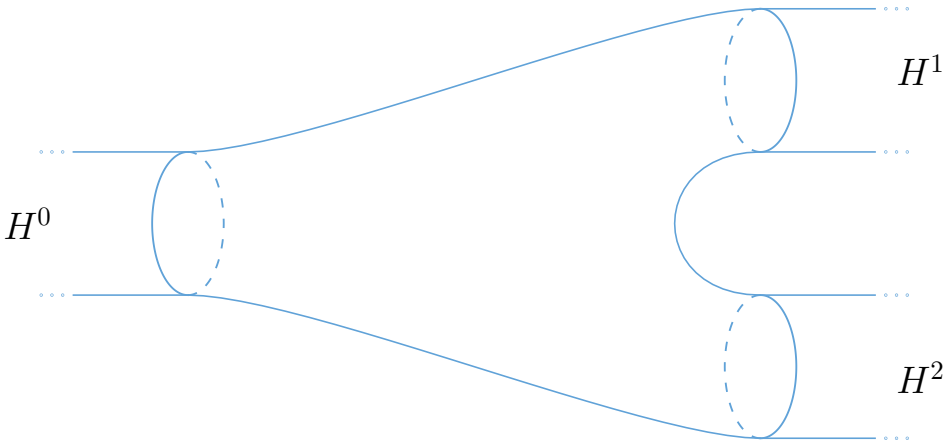
$$\int_P ||Du - \alpha \otimes X_{H^P}||^2 < \infty,$$

which satisfy the Cauchy-Riemann equation

$$J^P_z(Du(\xi) - \alpha(\xi) \cdot X_{H^P}) = Du(j\xi) - \alpha(j\xi) \cdot X_{H^P} \tag{3.2.2}$$

for every tangent vector $\xi \in T_zP$. Furthermore, the maps u need to converge along the end ε_i to the orbit x_i, i.e.

$$\lim_{|z| \to +\infty} u(\varepsilon_i(s,t) = x_i(t).$$



REMARK 3.2.3. The construction of a Cauchy-Riemann equation on P can be generalised to arbitrary Riemannian surfaces with at least one negative end and an arbitrary number of positive ends, see [Sch95, Sec. 2.3].

The final product

$$* : SH^i(T^*M;\sigma^E) \otimes SH^j(T^*M;\sigma^E) \to SH^{i+j}(T^*M;\sigma^E)$$

will then involve counting such pairs of pants between Hamiltonian orbits weighted by orientations.

3.2.a) Compactness of the Moduli Space of Pairs of Pants

Just like for the differential, we need to construct a compactification of $\mathcal{P}(x_0;x_1,x_2)$ in order to define operations on it. As is alluded to in [Abo14, Sec. 2.3.2] a suitable candidate is:

$$\overline{\mathcal{P}}(x_0;x_1,x_2) = \bigsqcup_{y_0,y_1,y_2} \overline{\mathcal{M}}(x_0;y_0) \times \mathcal{P}(y_0;y_1,y_2) \times \overline{\mathcal{M}}(y_1;x_1) \times \overline{\mathcal{M}}(y_2;x_2).$$

We will need the following result that is an adaptation of Gromov and Floer's compactness results on pseudoholomorphic equations on general Riemannian surfaces as for example discussed in [MS12, Ch. 4]:

LEMMA 3.2.4 ([Abo14, Lem. 2.3.8]). *If u_i is a sequence of elements of $\mathcal{P}(x_0;x_1,x_2)$, and there is a compact set of T^*M which contains the image of all the u_i's, then there is a subsequence converging to an element of $\overline{\mathcal{P}}(x_0;x_1,x_2)$.*

All that is left to prove compactness is thus an analogue of Lemma 2.3.7:

LEMMA 3.2.5 ([Abo14, Lem. 2.3.9]). *The image of every element of $\overline{\mathcal{P}}(x_0;x_1,x_2)$ is contained in the unit disc cotangent bundle.*

Proof (Sketch). The proof almost follows the proof of Lemma 2.3.7 verbatim. Assume that there exists a $u \in \overline{\mathcal{P}}(x_0;x_1,x_2)$ that exits the unit disc cotangent bundle. Choose a regular value $1+\varepsilon$ of $\rho\circ u$ and let Σ denote the inverse image of $[1+\varepsilon,+\infty)$ under $\rho\circ u$. Define $v := u_{|\partial\Sigma}$.

At this point we reap the rewards of having done the extra work of generalising the energy to arbitrary compact Riemannian surfaces in Section 2.6.a). The Assumption (3.2.1) implies the non-positivity of the differential 1-form, and thus we are in the case of Lemma 2.6.2, and can infer non-negativity of $E(v)$. At the same time, an application of Stoke's theorem implies that $E(v)$ is strictly negative, leading to a contradiction. □

3.2.b) Virtual Dimension of the Moduli Space of Pairs of Pants

In [Sch95, Ch. 2.4], Schwarz gives a complete description of the linearisation of the Floer operator on Riemannian surfaces with cylindrical ends. In our case, the linearisation of Equation (3.2.1) then is a map:

$$D_u : W^{1,p}(P,u^*(TT^*M)) \to L^p(P,u^*(TT^*M)) \otimes \Omega^{0,1}(P),$$

where $\Omega^{0,1}(P)$ is the vector bundle of anti-holomorphic 1-forms on P, and the Sobolev spaces are defined with respect to a metric on P which is cylindrical at the ends.

DEFINITION 3.2.6. *The virtual dimension of $\overline{P}(x_0; x_1, x_2)$ is the index of the operator D_u.*

A gluing technique similar to the one used in [Abo14, Ch. 1.4] can then be used to express the virtual dimension in terms of indices of the x_i. The most important results are the following:

PROPOSITION 3.2.7 ([Abo14, Prop. 2.3.13] and [Abo14, Lem. 2.3.14]). *The virtual dimension of the moduli space $\mathcal{P}(x_0; x_1, x_2)$ is*

$$\deg(x_0) - \deg(x_1) - \deg(x_2). \tag{3.2.3}$$

Furthermore, there is a canonical isomorphism of $\mathbb{Z}$-graded lines:

$$\det(D_u) \otimes \delta_{x_1}[w(x_1)] \otimes \delta_{x_2}[w(x_2)] \cong \delta_{x_0}[w(x_0)]. \tag{3.2.4}$$

$\square_{\text{ref.}}$

Moreover, standard transversality results, as for example presented in [FHS95], imply:

LEMMA 3.2.8 ([Abo14, Lem. 2.3.15]). *For generic data H^P and J^P, the moduli spaces $\mathcal{P}(x_0; x_1, x_2)$ are regular for all triples of orbits x_i, and this space is a smooth manifold of dimension*

$$\deg(x_0) - \deg(x_1) - \deg(x_2).$$

$\square_{\text{ref.}}$

3.2.c) Construction of the Product

For $i \in \{0, 1, 2\}$, let (H^i, J^i) be Floer data such that

$$H^1 + H^2 \leq H^0.$$

Assume moreover that the Floer data are generic, so that $(CF^*(H^i; \nu), d)$ is a cochain complex of every local system ν. At this point, we pose that $\nu := \sigma^E$, where E is a vector bundle on M.

Choose Floer data (H^P, J^P) over the pair of pants P as in the introduction of Section 3.2. In addition, we will assume that this data are regular, such that by Lemma 3.2.8 we have that $\mathcal{P}(x_0; x_1, x_2)$ has dimension 0 if $\deg(x_0) = \deg(x_1) + \deg(x_2)$. Thus $\det(D_u)$ is canonically trivial. Equation 3.2.3 then induces a map

$$*_u : o_{x_1}[w(x_1)] \otimes o_{x_2}[w(x_2)] \to o_{x_0}[w(x_0)]. \tag{3.2.5}$$

We now use the map of Equation (3.1.2) to add a local system

$$*_u \otimes \sigma^E_u : (o_{x_1} \otimes \sigma^E_{x_1}) \otimes (o_{x_2} \otimes \sigma^E_{x_2}) \to o_{x_0} \otimes \sigma^E_{x_0}. \tag{3.2.6}$$

By a slight abuse of notation, we will also write

$$*_u : CF^*(H^1; \sigma^E) \otimes CF^*(H^2; \sigma^E) \to CF^*(H^0; \sigma^E) \tag{3.2.7}$$

for the map induced on the cochain level.

Definition 3.2.9. *The pair of pants product*

$$* : CF^*(H^1;\sigma^E) \otimes CF^*(H^2;\sigma^E) \to CF^*(H^0;\sigma^E)$$

is the sum of the contributions of all rigid pairs of pants

$$* = \sum_{\substack{\deg(x_0)=\deg(x_1)+\deg(x_2)\\ u\in\mathcal{P}(x_0;x_1,x_2)}} *_u.$$

3.2.d) The Product Descends to Cohomology

Showing that the above structure also provides a product on cohomology boils down to the following proposition:

Proposition 3.2.10 ([Abo14, Prop. 2.3.18]). *The product $*$ satisfies*

$$d(a_1 * a_2) = da_1 * a_2 + (-1)^{\deg(a_1)} a_1 * da_2. \tag{3.2.8}$$

Before we go on to prove this result, we quickly show:

Corollary 3.2.11. *The map induced by $*$ is a map*

$$HF^*(H^1;\sigma^E) \otimes HF^*(H^2;\sigma^E) \to HF^*(H^0;\sigma^E).$$

Proof. Let $a_1 \in CF^i(H^1;\sigma^E), b_1 \in CF^{i-1}(H^1;\sigma^E)$ and $a_2 \in CF^j(H^2;\sigma^E)$ such that a_1 and a_2 are cycles. Then

$$\begin{aligned}(a_1 + db_1) * a_2 &= a_1 * a_2 + db_1 * a_2\\ &= a_1 * a_2 + d(b_1 * a_2) + (-1)^{\deg(b_1)+1} \underbrace{b_1 * da_2}_{=0}\\ &= a_1 * a_2 + d(b_1 * a_2).\end{aligned}$$

□

Proof of Proposition 3.2.10. Recall the Gromov-Floer compactification

$$\mathcal{P}(x_0;x_1,x_2) = \bigsqcup_{x_0',x_1',x_2'} \overline{\mathcal{M}}(x_0;x_0') \times \mathcal{P}(x_0';x_1',x_2') \times \overline{\mathcal{M}}(x_1;x_1') \times \overline{\mathcal{M}}(x_2;x_2').$$

Under the above regularity assumptions on the Floer data, we have that if

$$\deg(x_0) = \deg(x_1) + \deg(x_2) + 1,$$

then the space $\overline{\mathcal{P}}(x_0;x_1,x_2)$ is a manifold with boundary of dimension 1. The boundary strata are given by

$$\begin{aligned}&\bigsqcup_{\deg(x_0')+1=\deg(x_0)} \mathcal{M}(x_0;x_0') \times \mathcal{P}(x_0';x_1,x_2),\\ &\bigsqcup_{\deg(x_1)+1=\deg(x_1')} \mathcal{P}(x_0;x_1',x_2) \times \mathcal{M}(x_1';x_1),\\ &\bigsqcup_{\deg(x_2)+1=\deg(x_2')} \mathcal{P}(x_0;x_1,x_2') \times \mathcal{M}(x_2';x_2).\end{aligned}$$

This means that the trajectories effectively break up at the cylindrical ends. For degree reasons, there is only ever room for one more critical point, and thus we end up with the above distinct boundary strata.

We algebraically identify the first of these moduli spaces as the composition $d(x_1 * x_2)$ and the other two as $dx_1 * x_2$ and $x_1 * dx_2$ respectively. Just like for the differential, we have that the number of elements in the boundary of a 1-dimensional manifold vanishes modulo 2. It follows that $*$ is a chain map if we reduce to $\mathbb{Z}_2$-coefficients.

In order to prove the general case, we need to take orientations into account. The product of Equation (3.2.5)

$$*_u : o_{x_1}[w(x_1)] \otimes o_{x_2}[w(x_2)] \to o_{x_0}[w(x_0)]$$

introduces a sign of parity of

$$w(x_2) \cdot |x_1| \tag{3.2.9}$$

by the Koszul sign convention. This is due to the fact that on a pair of pants orientations on two of the three circles of a pair of pants determine an orientation on the remaining circle[4], thus $w(x_0) = w(x_1) + w(x_2) \mod 2$. In addition, the differential of Equation (2.5.4) introduces a sign

$$dS^{w(x_1)}\alpha := (-1)^{w(x_1)} S^{w(x_0)} d\alpha$$

and thus the various compositions of d and $*$ introduce the following signs:

$$\begin{array}{ccc} d(a_1 * a_2) & da_1 * a_2 & a_1 * da_2 \\ w(x_1) + w(x_2)(1 + |x_1|) & w(x_1) + w(x_2)(1 + |x_1|) & w(x_2)(1 + |x_1|). \end{array} \tag{3.2.10}$$

To each element $w \in \mathcal{P}(x_0; x_1, x_2)$ we can associate the following natural map from the gluing discussion in [Abo14, Eq. 2.3.22]:

$$\det(D_w) \otimes \det(D_{\psi_1}) \otimes \det(D_{\psi_2}) \cong \det(D_{\psi_0}). \tag{3.2.11}$$

This is the analogue of Proposition 2.4.9. We now discuss the sign of $a_1 * da_2$ in particular, the other signs follow in a similar manner. The product map $*$ in $a_1 * da_2$ is induced by the sum of isomorphisms

$$\det(D_u) \otimes \det(D_{\psi_1}) \otimes \det(D_{\psi_{x_2'}}) \cong \det(D_{\psi_0}) \tag{3.2.12}$$

for elements $u \in \mathcal{P}(x_0; x_1, x_2')$. At the same time, the differential is induced by the sum of maps

$$\det(D_v) \otimes \det(D_{\psi_{x_2}}) \cong \det(D_{\psi_{x_2'}}) \tag{3.2.13}$$

with $v \in \mathcal{M}(x_2'; x_2)$. Substituting (3.2.13) into (3.2.12), we obtain:

$$\det(D_u) \otimes \det(D_{\psi_{x_1}}) \otimes \det(D_v) \otimes \det(D_{\psi_{x_2}}) \cong \det(D_{\psi_{x_0}}). \tag{3.2.14}$$

[4]Recall that a pair of pants retracts onto the figure 8.

In a vein similar to the one presented in Proposition 2.5.9, we can glue the homotopy classes of u and v for some large gluing parameter to obtain an element w that is a suitable candidate for Equation (3.2.11). We still need to transpose $\det(D_{\psi_{x_1}}) \otimes \det(D_v)$, which introduces a sign whose parity is given by

$$|x_1|,$$

using the fact that this is the index of $D_{\psi_{x_1}}$, and $\det(D_{\psi_{x_1}})$ is supported in degree $\mathrm{ind}(D_{\psi_{x_1}})$. A similar analysis leads to the signs

$$\begin{array}{ccc} d(a_1 * a_2) & da_1 * a_2 & a_1 * da_2 \\ 1 & 0 & |x_1|, \end{array}$$

where the non-trivial parity in the case of $d(a_1 * a_2)$ comes from the fact that the translation vector field gives rise to an inward pointing vector in this situation. Combining this with the signs originating from the product and differential of Equation (3.2.10) we obtain:

$$\begin{array}{ccc} d(a_1 * a_2) & da_1 * a_2 & a_1 * da_2 \\ 1 + w(x_1) + w(x_2)(1 + |x_1|) & w(x_1) + w(x_2)(1 + |x_1|) & |x_1| + w(x_2)(1 + |x_1|). \end{array}$$

Since we only care about the relation between the signs, we can add an overall sign of $w(x_1) + w(x_2)(1 + |x_1|)$ to each sign to get:

$$\begin{array}{ccc} d(a_1 * a_2) & da_1 * a_2 & a_1 * da_2 \\ 1 & 0 & |x_1| + w(x_1) \end{array}$$

as our final signs of parity. Thus

$$(-1)^1 d(a_1 * a_2) + (-1)^0 da_1 * a_2 + (-1)^{|x_1|+w(x_1)} a_1 * da_2 = 0$$

which is exactly Equation (3.2.8). □

The next lemma concerns itself with the Independence of the product from the choice of Floer data on the pair of pants.

Lemma 3.2.12 ([Abo14, Lem. 2.3.19.]). *The map induced by $*$ on Floer cohomology*

$$HF^*(H^1; \sigma^E) \otimes HF^*(H^2; \sigma^E) \to HF^*(H^0; \sigma^E)$$

is independent of the choice of Floer data (H^P, J^P).

Proof (Sketch). Since the space of Floer data on the 3-punctured sphere is path connected, given two different choices of Floer data, we may connect them by a path. Following the mould of the proof of Lemma 2.6.9, the resulting cobordism shows that the maps induced at the two end points are homotopic, and thus induce the same map on cohomology. □

In order to show that the product extends to symplectic cohomology, we need the following lemma:

LEMMA 3.2.13 ([Abo14, Lem. 2.3.20.]). *Assume that $H^1 \leq K^1$, $H^2 \leq K^2$ and $K^1 + K^2 \leq H^0$. We have the commutative diagrams:*

$$\begin{array}{ccc} HF^*(H^1;\sigma^E) \otimes HF^*(H^2;\sigma^E) & \xrightarrow{\mathrm{id}\otimes\mathfrak{c}} & HF^*(H^1;\sigma^E) \otimes HF^*(K^2;\sigma^E) \\ \downarrow{\scriptstyle \mathfrak{c}\otimes\mathrm{id}} & \searrow^{*} & \downarrow{\scriptstyle *} \\ HF^*(K^1;\sigma^E) \otimes HF^*(H^2;\sigma^E) & \xrightarrow{*} & HF^*(H^0;\sigma^E) \end{array} \tag{3.2.15}$$

If in addition, $H^0 \leq K^0$, then we also have a commutative diagram:

$$\begin{array}{ccc} HF^*(H^1;\sigma^E) \otimes HF^*(H^2;\sigma^E) & \xrightarrow{*} & HF^*(H^0;\sigma^E) \\ & \searrow^{*} & \downarrow{\scriptstyle \mathfrak{c}} \\ & & HF^*(H^0;\sigma^E). \end{array} \tag{3.2.16}$$

COROLLARY 3.2.14 ([Abo14, Lem. 2.3.21.]). *The map $*$ induces a product*

$$SH^*(T^*M;\sigma^E) \otimes SH^*(T^*M;\sigma^E) \to SH^*(T^*M;\sigma^E).$$

Proof. The triangle (3.2.16) implies that the composition

$$HF^*(H^1;\sigma^E) \otimes HF^*(H^2;\sigma^E) \to HF^*(H^0;\sigma^E) \to SH^*(T^*M;\sigma^E) \tag{3.2.17}$$

is independent of all choices. Given Hamiltonians H^0, H^1 and H^2 such that $H^1 + H^2 \leq H^0$ and K^0 such that $H^0 \leq K^0$, we have that given $a_1 \otimes a_2 \in HF^*(H^1;\sigma^E) \otimes HF^*(H^2;\sigma^E)$ such that

$$\begin{aligned} a_1 *_{H^0} a_2 &= b_1 \in HF^*(H^0;\sigma^E) \\ a_1 *_{K^0} a_2 &= b_2 \in HF^*(K^0;\sigma^E) \end{aligned}$$

then by Diagram (3.2.16), we have $c(b_1) = b_2$ and thus $b_1 \sim b_2$ in the filtered colimit $SH^*(T^*M;\sigma^E)$.

On the other hand, the upper right triangle of (3.2.15) implies that the map (3.2.17) factors as

$$HF^*(H^1;\sigma^E) \otimes HF^*(H^2;\sigma^E) \to HF^*(H^1;\sigma^E) \otimes SH^*(T^*M;\sigma^E) \to SH^*(T^*M;\sigma^E).$$

Indeed, we have that

$$\varinjlim_{H^2} (HF^*(H^1;\sigma^E) \otimes HF^*(H^2;\sigma^E), \mathrm{id} \otimes \mathfrak{c}) = HF^*(H^1;\sigma^E) \otimes SH^*(T^*M;\sigma^E).$$

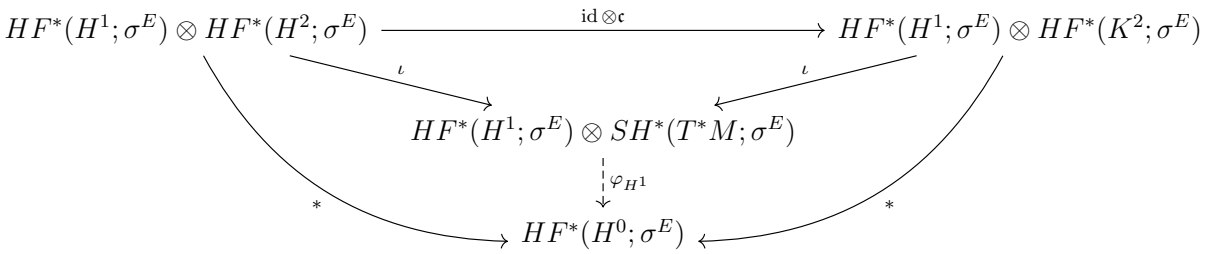
In addition, the diagram

$$\begin{array}{ccc} HF^*(H^1;\sigma^E) \otimes HF^*(H^2;\sigma^E) & \xrightarrow{\mathrm{id}\otimes\mathfrak{c}} & HF^*(H^1;\sigma^E) \otimes HF^*(K^2;\sigma^E) \\ & \searrow^{*} & \downarrow{\scriptstyle *} \\ & & HF^*(H^0;\sigma^E) \end{array}$$

commutes for all $H^1+H^2 \leq H^0$ by the upper triangle of (3.2.15). Thus this presents a cocone from the set of Hamiltonians to the Floer groups. The universal property of the colimit then implies that there exists a (unique) map

$$\varphi_{H^1} : HF^*(H^1;\sigma^E) \otimes SH^*(T^*M;\sigma^E) \to HF^*(H^0;\sigma^E)$$

such that



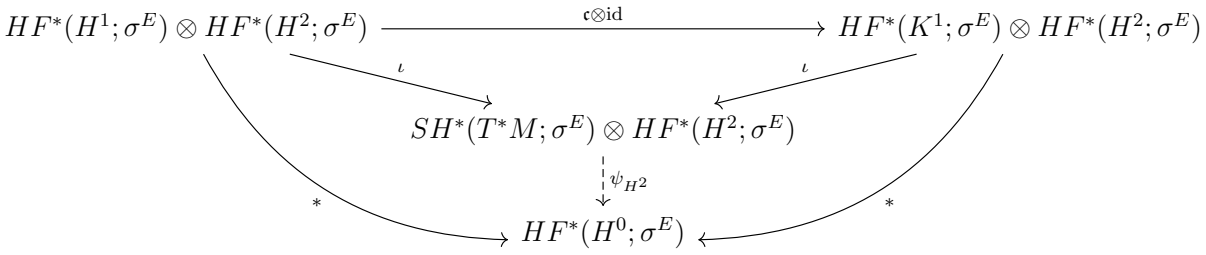
(3.2.18)

commutes. For an element $[a] \in SH^*(T^*M;\sigma^E)$ we will from now on denote a representative by $a \in HF^*(H;\sigma^E)$ for a suitable Hamiltonian H. Note that this diagram shows that φ_{H^1} can be calculated by choosing any representative for the equivalence class in $SH^*(T^*M;\sigma^E)$ and take the product with elements in $HF^*(H^1;\sigma^E)$.

We now want to extend this construction to a map $SH^*(T^*M;\sigma^E)\otimes SH^*(T^*M;\sigma^E) \to SH^*(T^*M;\sigma^E)$. First, observe that using the lower triangle of (3.2.15) yields a unique map

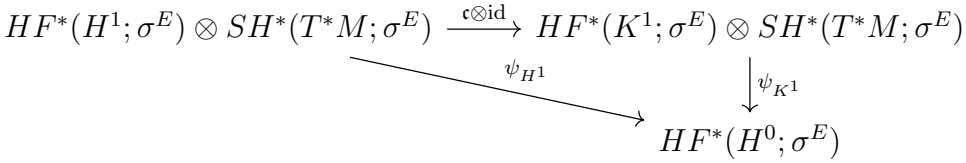
$$\psi_{H^2} : SH^*(T^*M;\sigma^E) \otimes HF^*(H^2;\sigma^E) \to HF^*(H^0;\sigma^E)$$

such that the following commutes



(3.2.19)

If we can show that



(3.2.20)

commutes, we obtain a unique map

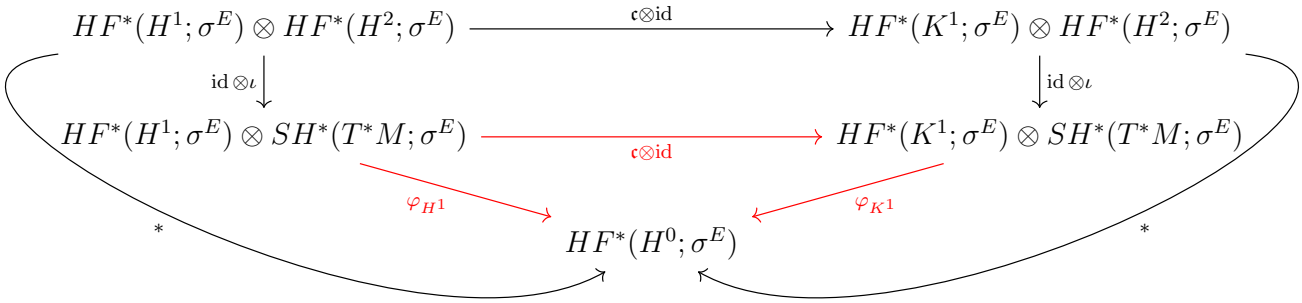
$$\mathrm{RL} : SH^*(T^*M;\sigma^E) \otimes SH^*(T^*M;\sigma^E) \to HF^*(H^0;\sigma^E)$$

by the universal property of the colimit. The notation RL encodes that we first perform the colimit construction on the right, and then on the left factor of $HF^*(H^1;\sigma^E)\otimes$

$HF^*(H^2;\sigma^E)$. To show commutativity, choose a generator $a\otimes[b]\in HF^*(H^1;\sigma^E)\otimes SH^*(T^*M;\sigma^E)$. Since φ_{H^1} is independent of the choice of representative, choose $b\in HF^*(H^2;\sigma^E)$ such that $\iota(b)=[b]$. The diagram

$$\begin{array}{ccc} HF^*(H^1;\sigma^E)\otimes HF^*(H^2;\sigma^E) & \xrightarrow{\mathfrak{c}\otimes\mathrm{id}} & HF^*(K^1;\sigma^E)\otimes HF^*(H^2;\sigma^E) \\ \downarrow{\scriptstyle \mathrm{id}\otimes\iota} & & \downarrow{\scriptstyle \mathrm{id}\otimes\iota} \\ HF^*(H^1;\sigma^E)\otimes SH^*(T^*M;\sigma^E) & \xrightarrow[\mathfrak{c}\otimes\mathrm{id}]{} & HF^*(K^1;\sigma^E)\otimes SH^*(T^*M;\sigma^E) \end{array} \tag{3.2.21}$$

then commutes by definition. Combining (3.2.21) with (3.2.20) and(3.2.18) we obtain



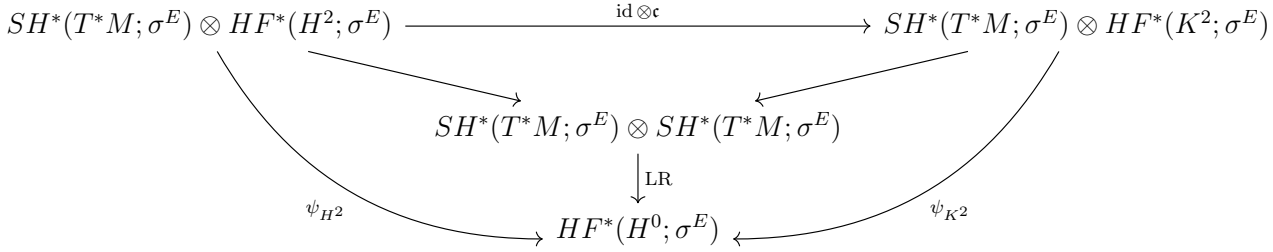
where everything except the red triangle commutes (a priori). The following calculation remedies this:

$$\begin{aligned} \varphi_{H^1}(a\otimes[b]) &= a*b \\ &= c(a)*b \\ &= \varphi_{K^1}(c(a)\otimes[b]) \\ &= (\varphi_{K^1}\circ(c\otimes\mathrm{id}))(a\otimes[b]). \end{aligned}$$

The only ambiguity left is that one can repeat the exact same construction by taking the colimit on the left of $HF^*(H^1;\sigma^E)\otimes HF^*(H^2;\sigma^E)$ first to obtain a unique map

$$LR : SH^*(T^*M;\sigma^E)\otimes SH^*(T^*M;\sigma^E)\to SH^*(T^*M;\sigma^E)$$

such that



commutes. But since all operations have been shown to be independent of the choice of representatives, and they all reduce to multiplication, we have

$$\begin{aligned}\mathrm{RL}([a]\otimes[b]) &= \varphi_{H^1}(a\otimes[b])\\ &= a * b\\ &= \psi_{H^2}([a]\otimes b)\\ &= \mathrm{LR}([a]\otimes[b]).\end{aligned}$$

□

3.2.e) Associativity and Commutativity

With this newly gained knowledge that the product is independent of the choice of representatives, we are able to reduce the proof of associativity and commutativity to the case of linear Hamiltonians.

Lemma 3.2.15 ([Abo14, Lem. 2.3.22.]). *Assume that we are given four Hamiltonians $H^1 + H^2 + H^3 \leq H^0$. Choose in addition two Hamiltonians $H^{1,2}$ and $H^{2,3}$ such that*

$$\begin{aligned}H^1 + H^2 + H^3 &\leq H^{1,2} + H^3 \leq H^0\\ H^1 + H^2 + H^3 &\leq H^1 + H^{2,3} \leq H^0.\end{aligned} \tag{3.2.22}$$

Then the following commutes:

$$\begin{array}{ccc} HF^*(H^1;\sigma^E)\otimes HF^*(H^2;\sigma^E) & \longrightarrow & HF^*(H^{1,2};\sigma^E)\otimes HF^*(H^3;\sigma^E)\\ \downarrow & & \downarrow\\ HF^*(H^{1,2};\sigma^E)\otimes HF^*(H^3;\sigma^E) & \longrightarrow & HF^*(H^0;\sigma^E). \end{array} \tag{3.2.23}$$

Proof (Sketch). This is a cobordism argument similar to the one of Lemma 2.6.9, and we are thus only going to show how one obtains a path linking the Floer data. Going through the diagram clockwise corresponds to counting configurations of two pairs of pants; the first pair linking orbits of H^2 and H^3 to those of H^0. Going counter-clockwise corresponds to the top side of Figure I.3

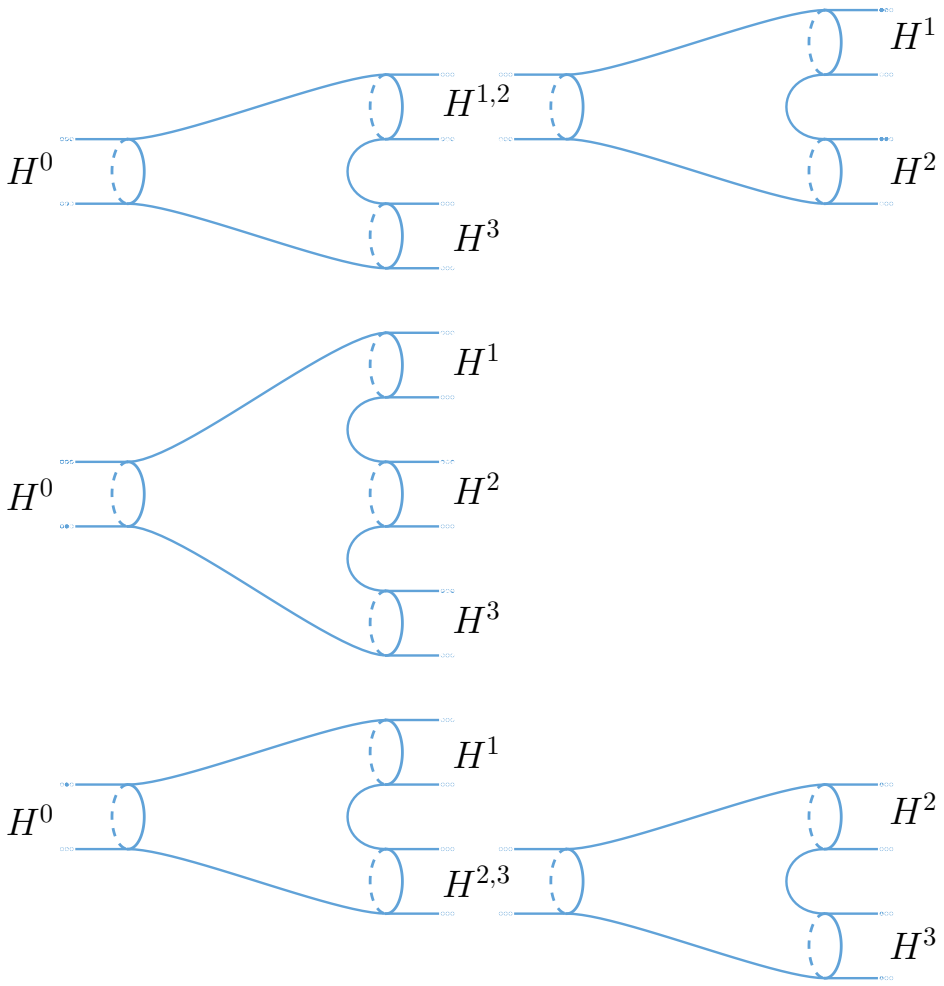


Figure I.3: Degenerations of 3-legged pair of pants.

Connected pairs of pants can be deformed into the 4-punctured sphere — or less abstractly a three-legged pair of pants. As the sources of the top and bottom of Figure I.3 lie on the boundary strata of the moduli space of complex structures on the three-legged pair of pants, and this space is connected, we have that we may choose a family of Floer data interpolating between the two configurations. The resulting cobordism then shows that the maps at the two ends are homotopic. □

Corollary 3.2.16. *The product on symplectic cohomology is associative.*

Proof. Let $[a], [b], [c] \in SH^*(T^*M; \sigma^E)$ be generators. Choose three Hamiltonians H^1, H^2 and H^3 such that there exist representatives $a \in HF^*(H^1; \sigma^E)$, $b \in HF^*(H^2; \sigma^E)$ and $c \in HF^*(H^3; \sigma^E)$ of $[a], [b]$ and $[c]$ respectively. In addition, choose $H^{1,2}, H^{2,3}$ and H^0 such that the assumptions of Equation (3.2.22) are

fulfilled. Then:

$$\begin{aligned}([a] * [b]) * [c] &= (a *_{(H^1,H^2;H^{1,2})} b) *_{(H^{1,2},H^3;H^0)} c \\ &\overset{(\spadesuit)}{=} a *_{(H^1,H2,3;H^0)} (b *_{(H^2,H^3;H^{2,3})} c) \\ &= [a] * ([b] * [c]),\end{aligned}$$

where (♠) follows from (3.2.23). □

Finally, we have the following commutativity result:

LEMMA 3.2.17 ([Abo14, Lem. 2.3.24.]). *The product induced by $*$ on symplectic cohomology satisfies:*

$$a_1 * a_2 = (-1)^{\deg(a_1)\deg(a_2)+w(a_1)w(a_2)} a_2 * a_1.$$

Proof (Sketch). The product is defined by fixing three points (z_0, z_1, z_2) on S^2 and counting maps on the 3-punctured sphere to T^*M, with appropriate asymptotic conditions at the cylindrical ends. The sphere S^2 admits a biholomorphism fixing z_0, and mapping z_1 to z_2 and vice-versa. Thus if we pull back the Floer data under this bihomolomorphism, we see that the product on cohomology is commutative up to a sign, since each element that appears in the product $a_1 * a_2$ also appears in the product of $a_2 * a_1$ (possibly with opposite sign).

In order to determine the sign, we need to recall Equation [Abo14, Eq. 2.3.22] which states that given a solution $u \in \mathcal{P}(x_0; x_1, x_2)$ we have:

$$\det(D_u) \otimes \det(D_{\psi^F_{x_1}}) \otimes \det(D_{\psi^F_{x_2}}) \cong \det(D_{\psi^F_{x_0}}).$$

Permuting $\det(D_{\psi^F_{x_1}})$ and $\det(D_{\psi^F_{x_2}})$ yields the sign

$$(-1)^{|x_1||x_2|}$$

by the Koszul sign convention. Note that the grading shift w introduces by Equation (3.2.9) an additional sign of $(-1)^{|a_1|w(a_2)}$ and $(-1)^{|a_2|w(a_1)}$ on both sides.

We conclude the proof via the following computation in $\mathbb{Z}_2$:

$$\begin{aligned}\deg(a_1)\cdot\deg(a_2) + w(a_1)w(a_2) &= (|a_1| - w(a_1))\cdot(|a_2| - w(a_2)) + w(a_1)w(a_2) \\ &= |a_1||a_2| - w(a_1)|a_2| - w(a_2)|a_1| + 2w(a_1)w(a_2) \\ &\overset{(\clubsuit)}{=} |a_1||a_2| + w(a_1)|a_2| + w(a_2)|a_1|,\end{aligned}$$

where (♣) followed by us working in $\mathbb{Z}_2$. □

3.3 The Unit

In this section, we show how "counting" Riemannian surfaces of genus 0 with one negative and no positive end, i.e. planes, allows us to construct a unit for our newly defined product on symplectic cohomology.

To this end, let (H, J) be generic data defining a Floer complex $CF^*(H;\nu)$. Choose a family H^z of linear Hamiltonians on T^*M parametrised by $z \in \mathbb{CP}^1 \setminus \{0\}$ which vanish near $\infty \in \mathbb{CP}^1$, and are given by $H^{e^{s+it}} = H_t$ whenever $s \ll 0$. Thus at the (negative) cylindrical end, the Hamiltonian shall only depend on the angular component. We require that the following monotonicity property holds:

$$H^z \leq H^{z'} \text{ whenever } |z'| \leq |z|. \tag{3.3.1}$$

In addition, choose a family J^z of almost complex structures such that $J^{e^{s+it}} = J_t$ for $s \ll 0$. Given an orbit x of H, let $\mathcal{P}(x)$ denote the space of maps

$$u : \mathbb{CP}^1 \setminus \{0\} \to T^*M \tag{3.3.2}$$

$$\lim_{s\to\infty} u(e^{s+it}) = x(t) \tag{3.3.3}$$

satisfying the Cauchy-Riemann equation:

$$(Du - X_{H^z} \otimes dt)^{0,1} = 0. \tag{3.3.4}$$

For the last equation, we used notation from complex geometry. Here $(Du - X_{H^z} \otimes dt)^{0,1} = 0$ means that the anti-holomorphic part should vanish, i.e. that

$$(Du - X_{H^z} \otimes dt) - J^z(Du - X_{H^z} \otimes dt)j = 0,$$

with j the complex structure on $\mathbb{CP}^1 \setminus \{0\}$.

Remark 3.3.1. Notice that by writing Equation (3.3.4) we are cheating a bit since the 1-form dt does not extend smoothly to $z = \infty$. However, since we assumed that H^z vanishes in a neighbourhood of said point, we can get away with this slight abuse of notation.

The monotonicity condition (3.3.1) serves the same role as Condition (3.2.1), and thus in a similar fashion as before, one can show that all elements of $\mathcal{P}(x)$ are contained in D^*M.

Note that $\mathcal{P}(x)$ is empty unless x is contractible; in which case $x^*(\det(TM))$ is orientable. To compute the virtual dimension in this case, observe that the unique trivialisation of $x^*(TT^*M)$ which extends to $u^*(TT^*M)$ is the one from Lemma 2.4.12. The linearisation of Equation (3.3.4) in such a trivialisation takes the form given in Equation (2.4.4). Thus we conclude that the virtual dimension is $\deg(x)$ and that we have a canonical isomorphism

$$\det(D_u) \cong \delta_x.$$

Applying the usual transversality arguments implies:

Lemma 3.3.2 ([Abo14, Lem. 2.4.3.]). *For generic choices of data (H^z, J^z), the moduli space $\mathcal{P}(x)$ is a smooth manifold of dimension* $\deg(x)$.

We now turn our attention to the case where the solution u is rigid, i.e. $\deg(x) = 0$. In this case, $\deg(D_u)$ is canonically trivial for every map $u \in \mathcal{P}(x)$, so we obtain a map

$$e_u : \mathbb{Z} \to o_x.$$

Considering that our final results will involve local coefficients, we tensor this map with the one of Equation (3.1.1) and compose the resulting map with the inclusion into the Floer complex to define the unit

$$\mathbb{Z} \to CF^*(H;\sigma^E).$$

By taking the sum over all rigid curves u, we obtain a map

$$e = \sum_{\substack{\deg(x)=0 \\ u\in\mathcal{P}(x)}} e_u. \tag{3.3.5}$$

In order to show that e descends to cohomology, let $\overline{\mathcal{P}}(x)$ be the Gromov-Floer space with codimension 1 stratum

$$\bigcup_x \overline{\mathcal{M}}(x,x') \times \overline{\mathcal{P}}(x').$$

When $\deg(x) = 1$, we have that $\overline{\mathcal{P}(x)}$ is a 1-dimensional manifold with boundary given by

$$\bigsqcup_{\deg(x')=\deg(x)+1} \mathcal{M}(x,x') \times \mathcal{P}(x').$$

This product corresponds to $d \circ e$. Since the boundary of a 1-dimensional oriented manifold consists of pairs of points of opposite orientation, we have that $d \circ e = 0$ and thus e descends to cohomology.

One can essentially repeat the proof of Lemma 2.6.9 to show that the map on cohomology

$$\mathbb{Z} \to HF^*(H;\sigma^E)$$

induced by e does not depend on the choice of Floer data H^z on the plane.

In order to further descend to symplectic cohomology, we need to show compatibility with the continuation maps:

Proposition 3.3.3. *Let $H^+ \leq H^-$ be two Hamiltonians and denote by $e^\pm : \mathbb{Z} \to HF^*(H^\pm;\sigma^E)$ their respective unit maps. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{e^+} & HF^*(H^+;\sigma^E) \\ & \searrow^{e^-} & \downarrow \mathfrak{c} \\ & & HF^*(H^-;\sigma^E). \end{array}$$

Proof. The following is an adaptation of [Abo14, Sec. 2.2.3.] to unit maps. Suppose we are given the following data:

- Floer data $(H_t^\pm, J_t^\pm)$ such that $H^+ \leq H^-$,
- families $(H_\pm^z, J_\pm^z)$ defining maps $e^\pm : \mathbb{Z} \to HF^*(H^\pm;\sigma^E)$,
- continuation map data $(H_{s,t}, J_{s,t})$ defining a map $\mathfrak{c} : HF^*(H^+;\sigma^E) \to HF^*(H^-;\sigma^E)$.

We then choose a family of Hamiltonian H_r^z on the plane parametrised by $r \in \mathbb{R}$ and $z \in \mathbb{C}\mathrm{P}^1 \setminus \{0\}$ subject to the following constraints:

1. For each r, there exists $S_r \geq 0$ such that
$$H_r^{e^{s+it}} = \begin{cases} 0 & \text{if } s \geq S_r \\ H_t^- & \text{if } s \leq -S_r. \end{cases}$$
2. For fixed r, we require that the monotonicity property holds:
$$H_r^z \leq H_r^{z'} \text{ when } |z'| \leq |z|.$$
3. If $r \ll 0$, then H_r^z is obtained by concatenating $H_{s,t}$ and H_+^z.
4. If $r \gg 0$, then H_r^z should agree with H_-^z.

Note that the notion of concatenation here is a slight variation on the one used in Proposition 2.6.10. We now glue the punctured plane and the cylinder: recall that if $s \gg 0$, then $H_{s,t}$ agrees with H_t^+ and that if $s \ll 0$ H_+^z also agrees with H_t^+. Thus choose $K \geq 0$ such that $H_+^{e^{-K+it}} = H_t^+$, $H_{K,t} = H_t^+$ and $H_{-K,t} = H_t^-$. We can now glue the cylinder and the plane as depicted in Figure I.4.

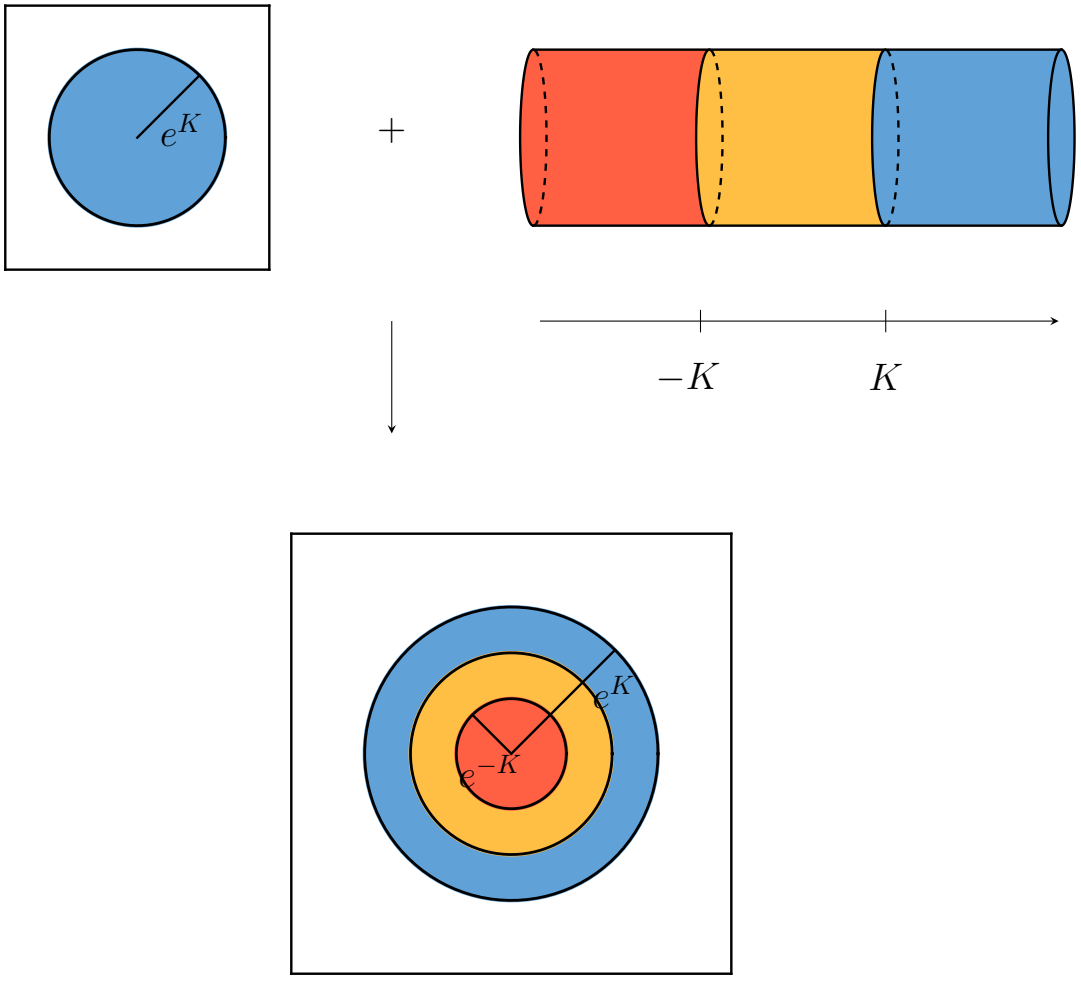


Figure I.4: Gluing of plane and cylinder.

For this we use the bijection

$$\{e^{s+it} \in \mathbb{C}\mathrm{P}^1 \setminus \{0\} \mid s \leq K\} \to (-\infty, K] \times [0,1] \subset Z$$
$$e^{s+it} \mapsto (s,t).$$

We obtain a glued Hamiltonian

$$G^z = \begin{cases} H^z_+ & \text{for } |z| \geq e^K \\ H_{s,t} & \text{for } |z| \leq e^K. \end{cases}$$

For $|z| \leq e^{-K}$, we have that $G^z = H_{s,t} = H^-_t$ since $s \leq -K$, and so we have that G^z is a suitable candidate to define a unit map for $HF^*(H^-; \sigma^E)$.

Since conditions 1) – 4) are local and convex in Z, we can produce an explicit family as follows: for some $R \geq 0$, choose a partition of unity (χ_+, χ_-) of $\mathbb{R}$ subordinate to the cover $(-R, +\infty)$ and $(-\infty, R)$. We define

$$H^z_r = \chi_+(r) G^z + \chi_-(r) H^z_-.$$

Next, we choose a family J^z_r of almost complex structures which are convex near S^*M that fulfils the following constraints:

- For fixed r, $J^{e^{s+it}} = J_t$ for $s \ll 0$.
- If $r \ll 0$, then J^z_r is obtained by concatenating J^z_+ and $J_{s,t}$.
- If $r \gg 0$, then $J^z_r = J^z_-$.

Given the families (H^z_r, J^z_r) and an orbit x_- of H^-, we define $\widetilde{\mathcal{P}}(x_-$ to be the moduli space of pairs (r,u) solving the differential equation:

$$(Du - X_{H^z_r} \otimes dt)^{0,1} = 0$$

and satisfying the asymptotic conditions

$$\lim_{s \to -\infty} u(s,t) = x_-(t).$$

By construction, the Gromov compactification of $\widetilde{\mathcal{P}}(x_-)$ admits two boundary strata, corresponding to the limits $r \to \pm\infty$, which are the disjoint union of products

$$\bigsqcup_{x_+ \in \mathcal{O}(H^+)} \mathcal{K}(x_-; x_+) \times \mathcal{P}(x_+) \quad \text{and}$$
$$\bigsqcup_{x_- \in \mathcal{O}(H^-)} \mathcal{P}(x_-).$$

If we assume that $\deg(x_-) = 0$, these strata respectively correspond to the composition $\mathfrak{c} \circ e^+$ and e^-. Applying the same method as in the proof of Proposition 2.6.10 we conclude the proof. □

Next, we show that our choice of terminology as the unit was indeed justified.

PROPOSITION 3.3.4. *Given three Hamiltonians such that $H^1 + H^2 \leq H^0$, the following commutes:*

$$\begin{array}{ccc} HF^*(H^1;\sigma^E) & \xrightarrow{\mathrm{id}\otimes e} & HF^*(H^1;\sigma^E)\otimes HF^*(H^2;\sigma^E) \\ & \searrow^{\mathfrak{c}} & \downarrow * \\ & & HF^*(H^0;\sigma^E). \end{array} \tag{3.3.6}$$

Proof (Sketch). This proof presents a lot of similarities to the previous proof, and thus we are only going to allude to how one obtains the necessary cobordism. Given orbits x_i of H^i for $i \in \{0,1\}$ of the same index, we can define a cobordism between the moduli spaces

$$\overline{\mathcal{K}}(x_0;x_1) \quad \text{and} \quad \bigsqcup_{\substack{x_2\in\mathcal{O}(H^2)\\ \deg(x_2)=0}} \mathcal{P}(x_0;x_1,x_2)\times\mathcal{P}(x_2),$$

corresponding to $\mathfrak{c}$ and $*\circ(\mathrm{id},e)$ respectively. The key insight is that gluing a plane to a leg of a pair of pants plugs a hole and thus we obtain a cylinder. This means that we can glue elements of $\mathcal{P}(x_2)$ and $\mathcal{P}(x_0;x_1,x_2)$ to obtain a solution to an equation on the cylinder, thus providing the cobordism. □

COROLLARY 3.3.5. *The map induced by $\bullet * e$ on symplectic cohomology is the identity. Furthermore, left multiplication by e also provides the identity.*

Proof. The first assertion follows from the fact that $\mathfrak{c}$ induces the identity on symplectic cohomology together with the commutativity of Diagram (3.3.6). The second assertion is a consequence of the commutativity of the product (up to sign) together with the fact that $w(e) = 0$ since all the loops that are involved in the definition of e must be contractible; and thus orientable. Considering in addition that $\deg(e) = 0$, we are able to conclude. □

4 Finite Model of the Free Loop Space

Viterbo's theorem concerns itself with relating the symplectic cohomology of T^*M to the homology of the free loop space. However, dealing with the latter groups turns out to be quite complicated, and thus we introduce a finite model for it. We then move on to define operations on the loop homology for which we will use Morse theory.

The construction that we will present stems from [Abo14, Sec.3], and the ideas presented therein relate in a straight forward manner to the original discussion of loop homology given in [CS99], who studied the case of oriented manifolds. The generalisation of the loop product to the non-orientable case was then done by Cohen and Jones in [CJ02] and by Laudenbach in [Lau11].

4.1 Introduction: the Space of Piecewise Geodesics

For the rest of this chapter, fix a metric on M such that

(H1) the injectivity radius is larger than 4.

Note that the existence of a global injectivity radius is guaranteed by our standing assumption that M is compact[5].

For each integer r together with a collection of strictly positive real numbers $\delta^r = \{\delta_r^1, \dots, \delta_r^r\}$ bounded by 2, we define a finite-dimensional approximation of the loop space as follows: for $1 \leq i \leq r-1$, consider

$$\begin{aligned} \rho_i : M^r &\to \mathbb{R} \\ (q_0, \dots, q_{r-1}) &\mapsto d(q_i, q_{i+1}), \end{aligned}$$

where d is the distance on M, and we use the convention $q_r := q_0$.

Definition 4.1.1. *The subset $\mathcal{L}^r_{\delta^r} M$ is the subset of M^r defined by the equations*

$$\rho_i \leq \delta_i^r \quad \text{for} \quad i = 1, \dots, r.$$

There exists a natural map

$$\text{geo} : \mathcal{L}^r_{\delta^r} M \to \mathcal{L}M$$

which sends a collection of points to the piecewise geodesic connecting them, parametrised at unit speed. Whenever $\delta_i^r \leq \delta_{i+1}^{r+1}$ for every integer i between 1 and r, we also have an inclusion

$$\iota : \mathcal{L}^r_{\delta^r} M \to \mathcal{L}^{r+1}_{\delta^{r+1}} M \tag{4.1.1}$$

$$(q_0, \dots, q_{r-1}) \mapsto (q_0, q_0, \dots, q_{r-1}). \tag{4.1.2}$$

Lemma 4.1.2. *Assume that $\delta_j^r \leq \delta_i^{r+1}$ for every pair (i, j). The map*

$$(q_0, \dots, q_{r-1}) \mapsto (q_0, \dots, q_{k-1}, q_k, q_k, q_{k+1}, \dots, q_{r-1}) \tag{4.1.3}$$

defines an inclusion $\mathcal{L}^r_{\delta^r} M \to \mathcal{L}^{r+1}_{\delta^{r+1}} M$. Furthermore, this map is homotopic to

$$\iota(q_0, \dots, q_{r-1}) \mapsto (q_0, q_0, \dots, q_{r-1}).$$

[5] And thus complete.

Proof. The only obstruction to the above map being an inclusion might be that the image is indeed contained in the target space. Keeping in mind that $\delta_j^r \leq \delta_i^{r+1}$ for every pair (i,j), we have

$$\rho_i(q_0,\dots,q_{k-1},q_k,q_k,q_{k+1},\dots,q_{r-1}) = \begin{cases} d(q_i,q_j) \leq \delta_i^r \leq \delta_i^{r+1} & \text{for } i<k, \\ d(q_k,q_k) = 0 \leq \delta_k^{r+1} & \text{for } i=k, \\ d(q_{i-1},q_i) \leq \delta_{i-1}^r \leq \delta_i^{r+1} & \text{for } i>k. \end{cases}$$

This shows that the map (4.1.3) defines an inclusion $\mathcal{L}^r_{\delta^r}M \to \mathcal{L}^{r+1}_{\delta^{r+1}}M$. In order to show that these maps are homotopic to ι, we are going to proceed by induction on k. For $k=1$, we have the map

$$(q_0,\dots,q_{r-1}) \mapsto (q_0,q_1,q_1,q_2,\dots,q_{r-1}).$$

Since $d(q_0,q_1) \leq \delta_r^r \leq 2$ which is smaller than the injectivity radius, there exists a geodesic $F:[0,1]\to M$ such that $F(0)=q_0$ and $F(1)=q_1$ connecting them. As F is a geodesic, we have that $d(q_0,F(t)) \leq d(q_0,q_1)$ for all t and thus $(q_0,F(t),q_1,\dots,q_{r-1})$ is contained in $\mathcal{L}^{r+1}M$ for all t, thus providing a homotopy to ι. The induction step then follows in exactly the same manner. □

Note that since we defined the element of the image of geo to be parametrised at unit speed, we have the commutative diagram:

$$\begin{array}{ccc} \mathcal{L}^{r-1}_{\delta^{r-1}}M & \xrightarrow{\iota} & \mathcal{L}^r_{\delta^r}M \\ & \searrow^{\text{geo}} & \downarrow{\scriptstyle \text{geo}} \\ & & \mathcal{L}M. \end{array} \tag{4.1.4}$$

Let us define $|\delta^r| = \sum_{i=1}^r \delta_i$. For each positive integer r, choose a sequence $\delta^r = \{\delta_1^r,\dots,\delta_r^r\}$ such that

$$\lim_{r\to+\infty} |\delta^r| = +\infty \tag{4.1.5}$$

$$\delta_i^r = \delta_{i+1}^{r+1} \quad \text{for } 1 \leq i \leq r. \tag{4.1.6}$$

Here now comes the main reason for calling the space of piecewise geodesics finite approximations to the loop space:

Proposition 4.1.3 ([Abo14, Prop. 3.2.4.]). *Assuming (4.1.5) and (4.1.6), the inclusion of the finite dimensional approximations induces a homotopy equivalence*

$$\bigcup_{r=1}^{\infty} \mathcal{L}^r_{\delta^r}M \to \mathcal{L}M.$$

Proof. Let $\mathcal{L}_{|\delta^r|}M$ denote the subset of piecewise smooth loops whose length is bounded by $|\delta^r|$. Since $\lim_{r\to\infty}|\delta^r| = +\infty$, we have that $\mathcal{L}M$ is a union of the subsets $\mathcal{L}_{|\delta^r|}M$. For each r, the image of geo is contained in $\mathcal{L}_{|\delta^r|}M$ we will show that it is in fact a homotopy equivalence. There exists a projection

$$\pi_r : \mathcal{L}_{|\delta^r|}M \to \mathcal{L}^r_{\delta^r}M$$

which sends a loop of length bounded by r to the evaluation of its arc-length parametrisation at the points

$$\left\{0, \frac{\delta_1^r}{|\delta^r|}, \ldots, \frac{\delta_1^r + \cdots + \delta_{r-1}^r}{|\delta^r|}\right\}.$$

Since the map geo sends points of $\mathcal{L}^r_{\delta^r} M$ to geodesics parametrised by unit speed, we see that $\pi_r \circ \mathrm{geo} = \mathrm{id}_{\mathcal{L}^r_{\delta^r} M}$. In order to see that $\mathrm{geo} \circ \pi_r \cong \mathrm{id}_{\mathcal{L}_{|\delta^r|} M}$, we have to investigate what the fibre above a point $(q_0, \ldots, q_{r-1}) \in \mathcal{L}^r_{\delta^r} M$ for π_r looks like, since these are all the elements that get sent to the same image under a subsequent application of geo.

This fibre $\pi_r^{-1}(q_0, \ldots, q_{r-1})$ is equal to the product of spaces of paths (not necessarily geodesics) of length bounded by δ_i^r with fixed endpoints in $(q_0, \ldots, q_{r-1})$. If we restrict ourselves to one such space of paths joining q_i and q_{i+1}, we have that since the length of each path is bounded by δ_i^r, each path must retract on to the unique geodesic linking q_i and q_{i+1}. This follows from the fact that for a distance ε smaller than the injectivity radius, the exponential map is a diffeomorphism from the open ε-ball in $T_{q_i} M$ to $\{q \in M \mid d(q_i, q) \leq \varepsilon\}$, and thus this provides a simply-connected neighbourhood for our retract. Combining these homotopies for each i yields $\mathrm{geo} \circ \pi_r \cong \mathrm{id}_{\mathcal{L}_{|\delta^r|} M}$. □

Given a local system ν on $\mathcal{L}M$, we associate by restriction a local system on $\mathcal{L}^r_{\delta^r} M$ which we still denote by ν.

Corollary 4.1.4. *Assuming (4.1.5) and (4.1.6), there exists an isomorphism*

$$\varinjlim_r H_*(\mathcal{L}^r_{\delta^r} M; \nu) \to H_*(\mathcal{L}M; \nu).$$

Proof. Given Equation (4.1.6), we have a sequence of inclusion

$$\ldots \longrightarrow \mathcal{L}^{r-1}_{\delta^{r-1}} M \xrightarrow{\iota^{r-1}} \mathcal{L}^r_{\delta^r} M \xrightarrow{\iota^r} \mathcal{L}^{r+1}_{\delta^{r+1}} M \longrightarrow \ldots$$

and thus the (filtered) colimit reduces to

$$\varinjlim \mathcal{L}^r_{\delta^r} M = \bigcup_{r=1}^{\infty} \mathcal{L}^r_{\delta^r} M.$$

The result now follows from the fact that filtered colimits commute with homology together with Proposition 4.1.3. □

4.2 Manifold Structures

In order to use Morse theory, we will endow $\mathcal{L}^r_{\delta^r} M$ with the structure of a submanifolds with corners of M^r for a generic choice of δ^r.

Lemma 4.2.1. *Let X be a smooth manifold, and $f_i : X^n \to \mathbb{R}$ a finite collection of smooth functions labelled by a set I. For a generic choice of real numbers $\{\delta_i\}$, the intersection*

$$\bigcap_{i \in I} f_i^{-1}((-\infty, \delta_i])$$

is a submanifold with corners.

Proof. Fix $i \in I$. By Sard's theorem, we have that δ_i for a generic choice are regular values. Thus by the regular value theorem[6], we have that $f_i^{-1}(\delta_i)$ is a submanifold of codimension 1. The sublevel set $f_i((-\infty, \delta_i]))$ is in turn a submanifold with boundary $f_i^{-1}(\delta_i)$ as a careful consideration of the slice chart provided by the regular value theorem shows.

In order to endow $\bigcap_i f_i^{-1}((-\infty, \delta_i])$ with the structure of a submanifold with corners, note that the interior $\bigcap_i f_i^{-1}((-\infty, \delta_i))$ is an intersection of open subsets of X and thus a submanifold. Thus the only points that need extra care are the boundary points.

We will show this by induction. First, we explain why we can assume that the hypersurfaces $f_i^{-1}(\delta_i)$ intersect transversally. Let $k = 2$, since the notion of transversality is generic, we have that if $f_1^{-1}(\delta_1)$ and $f_2^{-1}(\delta_2)$ do not intersect transversally, we can find a regular value $\delta_1 + \varepsilon$ arbitrarily close to δ_1 by Sard's theorem such that $f_1^{-1}(\delta_1 + \varepsilon)$ and $f_2^{-1}(\delta_2)$ intersect transversally.

Given a point $y \in f_1^{-1}(\delta_1) \cap f_2^{-1}(\delta_2)$, we have that there exist charts (φ_1, U_1) and (φ_2, U_2), where U_1 and U_2 are open subsets of $[0, +\infty) \times \mathbb{R}^{n-1}$ for $f_1^{-1}((-\infty, \delta_1])$ and $f_2^{-1}((-\infty, \delta_2])$ respectively such that $y \in \varphi_1(U_1) \cap \varphi_2(U_2)$. Since we can assume that $f_1^{-1}(\delta_1)$ and $f_2^{-1}(\delta_2)$ intersect transversally, we have that $U_1 \cap \varphi_1^{-1}(\varphi_2(U_2))$ is diffeomorphic to an open subset of $[0, +\infty) \times [0, +\infty)$. Note that in the case of non-transversality, $U_1 \cap \varphi_1^{-1}(\varphi_2(U_2))$ could have been fully included inside of the boundary $\{0\} \times \mathbb{R}$, and thus no longer been an open set. This provides us with a chart with corners at the intersection points.

The induction step then proceeds in exactly the same manner, only intersecting hyperplanes with the boundaries of sets of the type $[0, +\infty)^l \times \mathbb{R}^{n-l}$. □

As an immediate consequence, we have:

COROLLARY 4.2.2. *For a generic choice of constants δ^r, the set $\mathcal{L}^r_{\delta^r} M$ is a smooth submanifold with corners of M^r.*

The following definition fixes some notation:

DEFINITION 4.2.3. *Assume that for each r a family δ^r is fixed so that $\mathcal{L}^r_{\delta^r} M$ is a smooth manifold, and*

$$0 < \delta_i^r \leq \delta_j^{r+1} < 2 \quad \text{if} \quad 1 \leq i \leq r \quad \text{and} \quad 1 \leq j \leq r+1. \tag{4.2.1}$$

Define

$$\mathcal{L}^r M := \mathcal{L}^r_{\delta^r} M.$$

[6] Which is really only a different characterisation of being a submanifold, see [AD14, Thm. A.1.1.].

4.3 Morse Theory

Given a local system ν, we shall now study a Morse complex which computes $H_*(\mathcal{L}^r M; \nu)$. In order to circumvent having to deal with edge cases originating from critical points on the boundary of our manifold with corners, we are going to impose that the gradient flow lines of the Morse function f^r on $\mathcal{L}^r M$ that we study satisfy the condition:

$$df^r(\mathrm{grad}(\rho_i))_{|\rho_i^{-1}(\delta_i^r)} > 0.$$

This ensures that the gradient flow points outwards at the boundary; indeed, we have that $\rho_i^{-1}(\delta_i^r)$ provides the boundary component corresponding to δ_i^r and $\mathrm{grad}(\rho_i)$ is a vector field pointing out of the boundary.

As is common in Morse theory, we will however concern ourselves with the *negative* gradient flow of f^r that we will denote by ψ_t^r. Note that this flow is globally defined for all $t \geq 0$ but only defined on a closed subset of $\mathcal{L}^r M$ for $t \leq 0$ since negative flow lines may originate at the boundary.

For each critical point y of f^r, we define the stable and unstable manifold by:

$$W^u(y) := \{q \in M \mid \lim_{t \to -\infty} \psi_t^r(q) = y\},$$
$$W^s(y) := \{q \in M \mid \lim_{t \to +\infty} \psi_t^r(q) = y\}.$$

LEMMA 4.3.1. *The closure of $W^u(y)$ is disjoint from $\partial \mathcal{L}^r M$.*

Proof. Since the flow lines of ψ_t^r point inwards, we have that $W^u(y) \cap \partial \mathcal{L}^r M = \varnothing$ for any critical point y of f^r. Next by [BH04, Cor. 6.27.], we have that

$$\overline{W^u(y)} = \bigcup_{\substack{x \in \mathrm{Crit}(f^r) \\ \mathrm{ind}(y) \leq \mathrm{ind}(x)}} W^u(x).$$

Thus the intersection $\overline{W^u(y)} \cap \partial \mathcal{L}^r M$ is empty. □

DEFINITION 4.3.2. *The Morse index of a critical point is*

$$\mathrm{ind}(y) = \dim_{\mathbb{R}}(W^u(y)),$$

and the orientation line is

$$o_y := |W^u(y)|.$$

The Morse assumption implies that $W^u(y)$ and $W^s(y)$ intersect transversally at a single point[7]. Note that for this theorem, we need $\mathcal{L}^r M$ to be compact, but this is in fact given since $\mathcal{L}^r M$ is defined as the intersection of closed sets, and thus compact as a closed subset of the compact space M^r. We thus have a decomposition:

$$T_y \mathcal{L}^r M \cong T_y W^u(y) \oplus T_y W^s(y).$$

In terms of orientation lines, this isomorphism translates to:

[7] This is the content of for example [BH04, Thm. 4.2.].

Lemma 4.3.3 ([Abo14, Lem. 3.3.4.]). *There is a canonical isomorphism:*

$$o_y \otimes |W^s(y)| \cong |T_y \mathcal{L}^r M|. \tag{4.3.1}$$

We say that f^r is Morse-Smale if all stable and unstable manifolds meet transversally, i.e. if $W^u(y_1) \pitchfork W^s(y_0)$ for all critical points y_0 and y_1 of f^r. Under this condition, the intersection $W^u(y_1) \cap W^s(y_0)$ is a submanifold of $\mathcal{L}^r M$. It consists of all points on the trajectories connecting y_1 to y_0.

Proposition 4.3.4 ([AD14, Prop. 2.2.2.]). *The group $\mathbb{R}$ of translations in time acts freely on $W^u(y_1) \cap W^s(y_0)$ by $s \cdot y = \varphi_s^r(y)$ if $y_1 \neq y_0$.*

Proof. The fact that this is indeed a group action is clear. If $y_0 \neq y_1$, then there is no critical point in $W^u(y_1) \cap W^s(y_0)$. Let $y \in W^u(y_1) \cap W^s(y_0)$, since y is not a critical point, $f(\varphi_s^r(y))$ is decreasing in s. Thus $\varphi_s^r(x) = \varphi_{s'}^r(x)$ implies $s = s'$. □

Definition 4.3.5. *If $y_0 \neq y_1$, the moduli space of gradient trajectories $\mathcal{T}(y_0; y_1)$ is the quotient*

$$(W^u(y_1) \cap W^s(y_0))/\mathbb{R}.$$

Note that since $\overline{W^u(y_1)}$ is disjoint from $\partial \mathcal{L}^r M$, $W^u(y_1) \cap W^s(y_0)$ most certainly is too.

Lemma 4.3.6 ([Abo14, Lem. 3.3.9.]). *If f^r is Morse-Smale, then $\mathcal{T}(y_0; y_1)$ is a smooth manifold of dimension*

$$\operatorname{ind}(y_1) - \operatorname{ind}(y_0) - 1.$$

Proof. The dimension of $W^u(y_1)$ is $\operatorname{ind}(y_1)$, and the codimension of $W^s(y_0)$ is $\operatorname{ind}(y_0)$, thus the dimension of the transverse intersection is $\operatorname{ind}(y_1) - \operatorname{ind}(y_0)$. Taking the quotient reduces the dimension by 1. □

Just like for Floer trajectories, we will be interested in the case where $\mathcal{T}(y_0; y_1)$ is a 0-dimensional manifold, i.e. when

$$\operatorname{ind}(y_0) = \operatorname{ind}(y_1) - 1.$$

We will call such trajectories in $\mathcal{T}(y_0; y_1)$ rigid.

Lemma 4.3.7 ([Abo14, p. 3.3.10.]). *Every rigid trajectory $\gamma \in \mathcal{T}(y_0; y_1)$ induces a canonical isomorphism*

$$\partial_\gamma : o_{y_1} \to o_{y_0}.$$

Proof. At every point along γ, we have an isomorphism

$$|\mathbb{R} \cdot \partial_s \gamma| \otimes |T_{\gamma(s)} \mathcal{L}^r M| \cong o_{y_1} \otimes |W^s(y_0)| \tag{4.3.2}$$

induced by the exact sequence

$$\mathbb{R} \cdot \partial_s \gamma \longrightarrow T_{\gamma(s)} W^u(y_1) \oplus T_{\gamma(s)} W^s(y_0) \longrightarrow T_{\gamma(s)} \mathcal{L}^r M.$$

Combining Equation (4.3.1) and (4.3.2) and using the identification of orientations of $T_{y_0}\mathcal{L}^r M$ and $T_{\gamma(s)}\mathcal{L}^r M$ by parallel transport along γ, we obtain

$$\begin{aligned} o_{y_1} \otimes |W^s(y_0)| &\cong |\mathbb{R} \cdot \partial_s \gamma| \otimes |T_{\gamma(s)}\mathcal{L}^r M| \\ &\cong |\mathbb{R} \cdot \partial_s \gamma| \otimes |T_{y_0}\mathcal{L}^r M| \\ &\cong |\mathbb{R} \cdot \partial_s \gamma| \otimes o_{y_0} \otimes |W^s(y_0)| \end{aligned}$$

implying $|\mathbb{R} \cdot \partial_s \gamma|^{-1} \otimes o_{y_1} \cong o_{y_0}$. Fixing the opposite of the usual orientation of $\mathbb{R}$, we obtain the desired isomorphism. □

Remark 4.3.8. Note that the isomorphism above depends, up to a global sign, only on the choice of an orientation on $\mathbb{R}$, which in turn induces a direction for the flow line γ. The choice made in the proof of the above lemma corresponds to considering *positive* gradient flow lines with their natural orientation, which makes it compatible with the conventions we adapted for our Floer theoretic considerations of Section 2.

We now possess the necessary ingredients to define Morse homology with local coefficients. Given a local system ν on $\mathcal{L}^r M$, define the Morse complex of f^r with coefficients in ν to be the direct sum

$$CM_k(f^r; \nu) := \bigoplus_{\mathrm{ind}(y_0)=k} o_{y_0} \otimes \nu_{y_0}.$$

The differential is given by

$$\begin{aligned} \partial : CM_k(f^r; \nu) &\to CM_{k-1}(f^r; \nu) \\ \partial_{|o_{y_1} \otimes \nu_{y_1}} &= \sum_{\substack{\mathrm{ind}(y_0)=\mathrm{ind}(y_1)-1 \\ \gamma \in \mathcal{T}(y_0; y_1)}} \partial_\gamma \otimes \nu_\gamma, \end{aligned}$$

where ν_γ is the parallel transport map along γ. In order to show that $\partial^2 = 0$, we are going to reduce to the case of closed manifolds. By Lemma 4.3.1 points of the unstable manifold $W^u(y_1)$ can not accumulate on the boundary. Thus the natural compactification $\overline{\mathcal{T}}(y_0; y_1)$ of the moduli space of trajectories from y_1 to y_0 does not intersect the boundary $\partial \mathcal{L}^r M$. The proof that $\partial^2 = 0$ then follows analogously to the compact case, which is for instance shown in [AD14, Ch. 3].

We will write

$$HM_*(\mathcal{L}^r M; \nu) := H_*(CM_k(f^r; \nu), \partial)$$

for the resulting homology groups. The following proposition is a generalisation of standard results to our present setting with local coefficients and we refer to [Abo14] for a proof.

Proposition 4.3.9 ([Abo14, Prop. 3.3.13.]). *Morse homology is independent of f^r, and canonically isomorphic to the (ordinary) homology of $\mathcal{L}^r M$ with coefficients in ν.* □ref.

We only introduced the spaces $\mathcal{L}^r M$ as a finite model for $\mathcal{L}M$. In order to translate this construction to Morse homology, we need to give a Morse theoretic model for the maps induced by the inclusions of Equation (4.1.1).

The following space will be used to link critical points of f^r and f^{r-1} of the same degree, and thus serve as an analogue for the continuation maps of Floer cohomology. Given critical points y_0 of f^r and y_1 of f^{r-1}, we define

$$\mathcal{T}_\iota(y_0; y_1) := \iota(W^u(y_1)) \cap W^s(y_0).$$

We can think of every element of $\mathcal{T}_\iota(y_0; y_1)$ as a *piecewise trajectory*:

$$\gamma : \mathbb{R} \to \mathcal{L}^r M$$

which maps $(-\infty, 0]$ to the composition of ι with a negative gradient flow line of f^{r-1} converging at $-\infty$ to y_1 and $[0, +\infty)$ to negative gradient flow line of f^r converging at $+\infty$ to y_0, with a matching condition at 0.

Lemma 4.3.10 ([Abo14, Lem. 3.3.16.]). *If f^{r-1} is fixed, then for a generic choice of function f^r, $\mathcal{T}_\iota(y_0; y_1)$ is a smooth manifold of dimension*

$$\mathrm{ind}(y_1) - \mathrm{ind}(y_0).$$

Proof. The transversality statement follows from Sard's theoerem. To compute the dimension, observe that the dimension of $W^u(y_1)$ is $\mathrm{ind}(y_1)$ and that the codimension of $W^s(y_0)$ is $\mathrm{ind}(y_0)$. □

Let us now assume that $\mathcal{T}_\iota(y_0; y_1)$ is a 0-dimensional manifold. We claim that every element $\gamma \in \mathcal{T}_\iota(y_0; y_1)$ induces a canonical map

$$\iota_\gamma : o_{y_1} \to o_{y_0}. \tag{4.3.3}$$

To define this map, note that the transversality assumption implies that the natural map

$$T_{\gamma(0)} W^u(y_1) \oplus T_{\gamma(0)} W^s(y_0) \cong T_{\gamma(0)} \mathcal{L}^r M$$

is an isomorphism. This induces an isomorphism at the level of orientation lines

$$o_{y_1} \otimes |W^s(y_0)| \cong |T_{\gamma(0)} \mathcal{L}^r M|.$$

Identifying the orientations of $T_{y_0} \mathcal{L}^r M$ and $T_{\gamma(0)} \mathcal{L}^r M$ by parallel transport along γ, and combining this with Equation (4.3.1) yields

$$o_{y_1} \cong o_{y_0}$$

and thus the desired isomorphism of Equation (4.3.3).

Given a local system ν on $\mathcal{L}M$ with pullbacks under geo to $\mathcal{L}^{r-1} M$ and $\mathcal{L}^r M$ that we will also denote by ν, the commutativity of Diagram (4.1.4) implies that γ induces a canonical isomorphism

$$\nu_\gamma : \nu_{\gamma_1} \to \nu_{\gamma_0}.$$

We then define a map

$$\iota : CM_k(f^{r-1};\nu) \to CM_k(f^r;\nu)$$
$$\partial_{|o_{y_1}\otimes\nu_{y_1}} = \sum_{\substack{\mathrm{ind}(y_0)=\mathrm{ind}(y_1)\\ \gamma\in\mathcal{T}_\iota(y_0;y_1}} \iota_\gamma\otimes\nu_\gamma.$$

In order to prove that ι is a chain map, we consider the space

$$\overline{\mathcal{T}}_\iota(y_0;y_1) := \iota(\overline{W^u}(y_1))\cap\overline{W^s}(y_0).$$

Since $\overline{W^u}(y_1)$ and $\overline{W^s}(y_0)$ are both closed sets of a compact set, they are compact. Thus since the image of a compact set is compact, we have that $\overline{\mathcal{T}}_\iota(y_0;y_1)$ is compact as the intersection of compact sets.

Proposition 4.3.11. *The map ι is a chain map.*

Proof (Sketch). Let y_1 and y_0 be critical points of f^{r-1} and f^r respectively such that

$$\mathrm{ind}(y_1) = \mathrm{ind}(y_0)+1.$$

Under our regularity assumptions, $\overline{\mathcal{T}_\iota}(y_0;y_1)$ is a manifold with boundary of dimension 1. By [Abo14, Lem. 3.3.14.], the boundary strata of $\overline{\mathcal{T}_\iota}(y_0;y_1)$ are given by

$$\bigsqcup_{\substack{y_0'\in\mathrm{Crit}(f^r)\\ \mathrm{ind}(y_0')=\mathrm{ind}(y_0)+1}} (\iota(\overline{W^u}(y_1))\cap W^s(y_0'))\times\mathcal{T}(y_0;y_0')$$
$$\bigsqcup_{\substack{y_1'\in\mathrm{Crit}(f^{r-1})\\ \mathrm{ind}(y_1)=\mathrm{ind}(y_1')+1}} \mathcal{T}(y_1';y_1)\times(\iota(\overline{W^u}(y_1'))\cap W^s(y_0)).$$

We algebraically interpret the first of these moduli spaces as $\partial\circ\iota$ and the second one as $\iota\circ\partial$. By the usual cobordism argument, we thus conclude that $\iota\circ\partial-\partial\circ\iota=0$. □

4.4 Operations on Loop Homology

In order to define an isomorphism from symplectic cohomology to loop homology in full generality, we need to introduce a local system η. Whenever M admits a Spin structure, this local system will be trivial and supported in degree $-n$.

The local system η is the tensor product of three local systems: the first one is the local system

$$\sigma := \sigma^{TM} \tag{4.4.1}$$

of trivialisations of the pullback of $TM\oplus 3\deg(TM)$ from the introduction of 3. We defined the fibre of σ at a loop γ to be the quotient of the free abelian group generated by these two stable trivialisations, by the relation that their sum vanishes. Recall that $|M|_q$ denotes the space of orientations of M at $q\in M$ and that $|M|_q^{-1}$ is the dual line to it carrying a canonical map

$$|M|_q\otimes|M|_q^{-1}\to\mathbb{Z}.$$

Thus by definition $|M|_q$ and $|M|_q^{-1}$ are supported in opposite degrees. We now have all necessary ingredients to define the fibre of η at a loop γ:

$$\eta_\gamma := \sigma_\gamma \otimes |M|^{-1}_{\gamma(0)} \otimes (|M|_{\gamma(0)}[n])^{\otimes -w(\gamma)}, \tag{4.4.2}$$

where $[n]$ indicates a *downward* shift of the degree by n, and $w(\gamma)$ is 0 or -1 depending on whether $\gamma^*(T^*M)$ is orientable or not. Since $|M|$ is naturally supported in degree n, the above local system is always supported in degree $-n$, independent of the orientability of $\gamma(T^*M)$.

Remark 4.4.1. We will never explicitly show that the structures that we are subsequently going to define actually satisfy the axioms of a ring structure. It is going to be the isomorphism of Section 6 that will imply that loop homology is indeed a ring.

4.4.a) The Unit

Observe that there exists a natural inclusion

$$\jmath : M \hookrightarrow \mathcal{L}M,$$

sending a point $q \in M$ to the constant loop at q. A local system on $\mathcal{L}M$ thus induces a local system on M by restricting to constant loops. If we apply this construction to η, we obtain an isomorphism of local systems

$$|M|^{-1} \xrightarrow{\sim} \jmath^*(\eta);$$

indeed, note that constant loops are orientable and that the local system σ of Equation (4.4.2) is trivial in this case. Combining this with the fact that $\mathcal{L}^1 M \cong M$, we obtain a map

$$HM_*(M; |M|^{-1}) \longrightarrow \varinjlim_r HM_*(\mathcal{L}^r M; \eta) \cong H_*(\mathcal{L}M; \eta).$$

Given a maximum y of a Morse function f^1 on M, we have that

$$T_y W^u(y) \cong T_y M.$$

Thus by the definition of o_y as the orientation line of $|W^u(y)|_y$, together with the definition of the dual line $|M|_y^{-1}$, there is an isomorphism

$$e_y : \mathbb{Z} \to |M|_y \otimes |M|_y^{-1} \cong o_y \otimes |M|_y^{-1}.$$

At the chain level, we define

$$\begin{aligned} e : \mathbb{Z} &\to CM_0(M; |M|^{-1}) \\ 1 &\mapsto \sum_{\mathrm{ind}(y)=n} e_y(1). \end{aligned}$$

Since every 0-chain is also a cycle, we define:

Definition 4.4.2. *The unit e of the loop space homology with coefficients in η is the composition*

$$\mathbb{Z} \longrightarrow HM_0(M; |M|^{-1}) \longrightarrow H_0(\mathcal{L}M; \eta).$$

4.4.b) The Loop Product with Coefficients in $|M|^{-1}$

In this section, we are going to use maps at the level of finite approximations to define a product

$$* : H_*(\mathcal{L}M; |M|^{-1}) \otimes H_*(\mathcal{L}M; |M|^{-1}) \to H_*(\mathcal{L}M; |M|^{-1}).$$

We will then in the following section discuss how one can extend this result to different coefficients. To start off, we define an analogue to the evaluation maps for $\mathcal{L}^r M$. Define

$$\mathrm{ev}_0 : \mathcal{L}^r M \to M$$

to be the projection to the first coordinate. In order to define the product, we will need to link different finite approximation spaces. For this, we now introduce the notion of a fibre product of submanifolds.

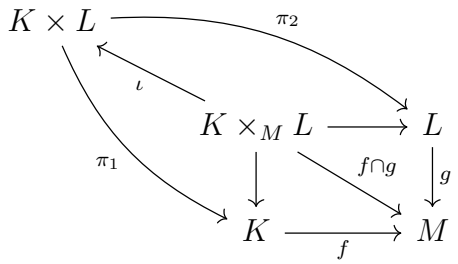
LEMMA 4.4.3. *If $K, L \subset M$ are two transverse embedded submanifolds, then $K \cap L$ is also a regular submanifold of dimension* $\dim K + \dim L - \dim M$.

Proof. Let $p \in K \cap L$. Then there is a neighbourhood U of p for which $K \cap U = f^{-1}(0)$ for 0 a regular value of some function $f : U \to \mathbb{R}^k$ and $L \cap U = g^{-1}(0)$ for 0 a regular value of a function $g : L \cap U \to \mathbb{R}^l$, where K and L both have codimension k and l respectively. Note that such functions always exist since we are dealing with embeddings. The assumptions imply that 0 must be a regular value of

$$(f, g) : K \cap U \to \mathbb{R}^{k+l};$$

indeed, $\ker(D(f,g)(q)) = \ker(Df(q)) \cap \ker(Dg(q)) = T_qK \cap T_qL$ for every $q \in (f,g)^{-1}(0) = K \cap L \cap U$. The space $T_qK \cap T_ql$ has codimension $k + l$. Thus $(f, g)|_{U \cap K \cap L}$ provides an appropriate slice chart for a point p in $K \cap L$. □

PROPOSITION 4.4.4. *If $f : K \to M$ and $g : L \to M$ are transverse smooth maps, then $K \times_M L := \{(a, b) \in K \times \mid f(a) = g(b)\}$ is a smooth manifold such that*



commutes, where ι is the inclusion map and $f \cap g : (a, b) \mapsto f(a) = g(b)$.

Proof. Consider the graphs $\Gamma_f \subset K \times M$ and $\Gamma_g \subset L \times M$. Then we show that the following intersection of regular submanifolds is transverse:

$$\Gamma_{f \cap g} = (\Gamma_f \times \Gamma_g) \cap (K \times L \times \Delta_M),$$

where $\Delta_M = \{(p, p) \in M \times M \mid p \in M\}$ is the diagonal. Let $f(k) = g(l) = m$ such that $x = (k, l, m, m) \in K \times L \times M \times M$, and note that

$$T_x(\Gamma_f \times \Gamma_g) = \{(v, Df(k)[v], (w, Dg(l)[w])) \mid v \in T_kK,\ w \in T_lL\}$$

and

$$T_x(K \times L \times \Delta_M) = \{((v,m),(w,m)) \mid v \in T_kK,\ w \in T_lL,\ m \in T_pM\}.$$

In order to show transversality of these spaces, we need to show that for any $(m_1, m_2) \in T_pM \times T_pM$, there exist $v' \in T_kK,\ w' \in T_lL$ and $m \in T_pM$ such that:

$$(Df(v') + m, Dg(w') + m) = (m_1, m_2).$$

By transversality, the m_i's decompose as $Df(v_i) + Dg(w_i)$ for some (v_i, w_i). Next, set $v' := v_2$ and $w' := w_1$, then

$$\begin{cases} Df(v_2) + m &= Df(v_1) + Dg(w_1), \\ Dg(w_1) + m &= Df(v_2) + Dg(w_2), \end{cases}$$

implying that $Df(v_1 - v_2) + Dg(w_1) = Df(v_2) + Dg(w_2 - w_1)$. Thus by setting $m := Df(v_1 - v_2) + Dg(w_1)$, we conclude that both spaces are transverse. Invoking Lemma 4.4.3 shows that $\Gamma_{f\cap g}$ is an embedded submanifold of $K \times L \times M \times M$. Since $\Gamma_{f\cap g} \subset K \times L \times \Delta_M$, we may see $\Gamma_{f\cap g}$ as an embedded submanifold of $K \times L \times M$ under the projection diffeomorphism $\Delta_M \to M$. Next, observe that the restriction of the projection onto $K \times L$ to the submanifold $\Gamma_{f\cap g}$ is an embedding with image $K \times_M L$. Hence $K \times_M L$ is a smooth manifold and $\Gamma_{f\cap g}$ may be viewed as the graph of a smooth map $f \cap g : K \times_M L \to M$, which implies that the above diagram commutes. □

Remark 4.4.5. We will call the above space $K \times_M L$ the fibre product of K and L over M. The reason for this nomenclature is that if $K := M \times Z_1$ and $L := M \times Z_2$ are two trivial fibre bundles over M, then $K \times_M L$ has fibres $Z_1 \times Z_2$ over any point of M. Furthermore, we will often suppress a lot of the involved maps and only provide the relevant maps and or spaces as subscripts.

We are now able to link different finite approximation spaces by taking the fibre product with the map ev_0. Fix an embedding

$$\begin{gathered} \mathcal{L}^{r_1}M \times_M \mathcal{L}^{r_2}M \to \mathcal{L}^{r_1+r_2}M \\ ((q_0, q_1, \dots, q_{r_1-1}), (q_0, q'_1, \dots, q'_{r_2-1})) \mapsto (q_0, q_1, \dots, q_{r_1-1}, q_0, q'_1, \dots, q'_{r_2-1}). \end{gathered} \tag{4.4.3}$$

This map is well-defined since we set $q_0 = q_{r_1} = q_{r_2}$ at the beginning, together with Equation (4.2.1).

Lemma 4.4.6. *The following diagram commutes up to homotopy:*

$$\begin{array}{ccc} \mathcal{L}^{r_1}M \times_{\mathrm{ev}_0} \mathcal{L}^{r_2}M & \longrightarrow & \mathcal{L}^{r_1+r_2}M \\ \downarrow & & \downarrow \\ \mathcal{L}^{r_1}M \times_{\mathrm{ev}_0} \mathcal{L}^{r_2+1}M & \longrightarrow & \mathcal{L}^{r_1+r_2+1}M. \end{array}$$

Furthermore, this diagram strictly commutes if we switch the roles of the first and the second factor.

Proof. The last assertion follows by chasing an element of $\mathcal{L}^{r_1}M \times_{\mathrm{ev}_0} \mathcal{L}^{r_2}M$ around the diagram. As for the first one, applying the counterclockwise portion of the diagram to an element $((q_0, q_1, \ldots, q_{r_1-1}), (q_0, q'_1, \ldots, q'_{r_2-1}))$ yields:

$$(q_0, q_1, \ldots, q_{r_1-1}, q_0, q_0, q'_1, \ldots, q'_{r_2-1}),$$

and going clockwise yields:

$$(q_0, q_0, q_1, \ldots, q_{r_1-1}, q_0, q'_1, \ldots, q'_{r_2-1}).$$

In an effort not to overload the letter ι used for the many inclusions, we will temporarily denote the inclusion of Lemma 4.1.2 by:

$$\jmath_k(q_0, \ldots, q_{r-1}) \mapsto (q_0, \ldots, q_{k-1}, q_k, q_k, q_{k+1}, \ldots, q_{r-1}).$$

Consequently, we notice that going counterclockwise equals

$$\jmath_{r_1}(q_0, \ldots, q_{r_1-1}, q_0, q'_1, \ldots, q'_{r_2-1})$$

which can be homotoped into

$$\iota(q_0, \ldots, q_{r_1-1}, q_0, q'_1, \ldots, q'_{r_2-1}).$$

□

Given critical points of y_0 of $f^{r_1+r_2}$ and y_i of f^{r_i}, we define

$$\mathcal{T}(y_0; y_1, y_2) := (W^u(y_1) \times_M W^u(y_2)) \times_{\mathcal{L}^{r_1+r_2}M} W^s(y_0). \tag{4.4.4}$$

This definition is rather contrived. The geometric picture one should have in mind is the one coming from the maps $\mathrm{geo} : \mathcal{L}^r M \to \mathcal{L}M$. These spaces will be finite approximations of the space of trajectories linking two loops that share a basepoint with a third loop. We can further unpack the definition:

Lemma 4.4.7 ([Abo14, Lem. 3.4.13]). *Assume $r_1 \neq r_2$. If f^{r_1} is fixed, we may choose f^{r_2} and $f^{r_1+r_2}$ such that the fibre product in Equation (4.4.4) is transverse. In this case, $\mathcal{T}(y_0; y_1, y_2)$ is a smooth manifold of dimension*

$$\mathrm{ind}(y_1) + \mathrm{ind}(y_2) - n - \mathrm{ind}(y_0).$$

Proof. First, choose f^{r_2} so that $\mathrm{ev}_0|_{W^u(y_2)}$ is transverse to $\mathrm{ev}_0|_{W^u(y_1)}$. Since the dimension of these manifolds is $\mathrm{ind}(y_1)$ and $\mathrm{ind}(y_2)$, the fibre product $W^u(y_1) \times_M W^u(y_2)$ has codimension

$$\mathrm{ind}(y_1) + \mathrm{ind}(y_2).$$

Using the map of Equation (4.4.3), we obtain an inclusion

$$W^u(y_1) \times_M W^u(y_2) \to \mathcal{L}^{r_1}M \times_{\mathrm{ev}_0} \mathcal{L}^{r_2}M \to \mathcal{L}^{r_1+r_2}M.$$

For a generic choice of Morse function $f^{r_1+r_2}$, the submanifold $W^s(y_0)$ is transverse to this inclusion. Since $\dim(W^s(y_0)) = n - \mathrm{ind}(y_0)$, we have that

$$\begin{aligned}\dim((W^u(y_1) \times_M W^u(y_2)) \times_{\mathcal{L}^{r_1+r_2}} W^s(y_0)) &= \mathrm{ind}(y_1) + \mathrm{ind}(y_2) - n + n - \mathrm{ind}(y_0) - n \\ &= \mathrm{ind}(y_1) + \mathrm{ind}(y_2) - \mathrm{ind}(y_0) - n.\end{aligned}$$

□

Remark 4.4.8. A small perturbation of the definition allows us to achieve transversality even if f^{r_1}, f^{r_2} and $f^{r_1+r_2}$ are all fixed. This is particularly important if we are interested in dropping the assumption that $r_1 \neq r_2$ in which case the fibre product $W^u(x) \times_M W^u(x)$ is never transverse unless x is a maximum.

The key insight consists of choosing vector fields X_s^2 on $\mathcal{L}^{r_2}\mathrm{M}$ and X_s^0 on $\mathcal{L}^{r_1+r_2}M$ parametrised by the interval, which integrate to diffeomorphisms $\psi_{2,X}$ and $\psi_{0,X}$. We define

$$\mathcal{T}^X(y_0; y_1, y_2) := (W^u(y_1) \times_M \psi_{2,X}) \times_{\mathcal{L}^{r_1+r_2}M} \psi_{0,X}^{-1}(W^s(y_0)).$$

Let us now consider the case where

$$\mathrm{ind}(y_0) = \mathrm{ind}(y_1) + \mathrm{ind}(y_2) - n.$$

This implies that $\mathcal{T}(y_0; y_1, y_2)$ is 0-dimensional.

Lemma 4.4.9. *[Abo14, Lem. 3.4.15.] To every element $\gamma \in \mathcal{T}(y_0; y_1, y_2)$, we can canonically associate a map*

$$*_\gamma : (o_{y_1} \otimes \mathrm{ev}_0^* |M|_{y_1}^{-1}) \otimes (o_{y_2} \otimes \mathrm{ev}_0^* |M|_{y_2}^{-1}) \to (o_{y_0} \otimes \mathrm{ev}_0^* |M|_{y_0}^{-1}). \tag{4.4.5}$$

Proof. By construction, we have a homotopy between $\mathrm{geo}(y_0)$ and the concatenations of $\mathrm{geo}(y_1)$ and $\mathrm{geo}(y_2)$. In particular, we have a canonical isomorphism

$$\mathrm{ev}_0^* |M|_{y_1}^{-1} \cong \mathrm{ev}_0^* |M|_{y_2}^{-1} \cong \mathrm{ev}_0^* |M|_{y_0}^{-1}.$$

Using this isomorphism, our target isomorphism (4.4.5) can be rewritten as

$$o_{y_1} \otimes \mathrm{ev}_0^* |M|_{y_1}^{-1} \otimes o_{y_2} \cong o_{y_0}, \tag{4.4.6}$$
$$o_{y_1} \otimes o_{y_2} \otimes \mathrm{ev}_0^* |M|_{y_1}^{-1} \cong o_{y_0}, \tag{4.4.7}$$
$$o_{y_1} \otimes o_{y_2} \cong o_{y_0} \otimes \mathrm{ev}_0^* |M|_{y_1}^{-1}. \tag{4.4.8}$$

As it turns out, it will be easier to show this last isomorphism. Fix $q \in M$ together with $z_i \in W^u(y_i)$ such that $z_i \in \mathrm{ev}_0^{-1}(q)$. Then by transversality, the following map is surjective:

$$\begin{aligned} T_{z_1}W^u(y_1) \oplus T_{z_2}W^u(y_2) &\to T_qM \\ (v_1, v_2) &\mapsto D\,\mathrm{ev}_0(z_1)[v_1] - D\,\mathrm{ev}_0(z_2)[v_2]. \end{aligned}$$

The crux of the argument is now that the kernel of this map can be identified with $T_{(z_1,z_2)}(W^u(y_1) \times_M W^u(y_2))$, and thus we have the short exact sequence

$$0 \longrightarrow T_{(z_1,z_2)}(W^u(y_1) \times_M W^u(y_2)) \longrightarrow T_{z_1}W^u(y_1) \oplus T_{z_2}W^u(z_2) \longrightarrow T_qM.$$

This induces an isomorphism of orientation lines

$$|T_(z_1, z_2)(W^u(y_1) \times_M W^u(y_2))| \otimes |T_qM| \to |T_{z_1}W^u(z_1)| \otimes |T_{z_2}W^u(z_2)|. \tag{4.4.9}$$

Since $\mathrm{ev}_0(z_1) = q$, we have that an orientation of T_qM is equivalent to an element of $(\mathrm{ev}_0 |M|)_{z_1}$. Using this together with the identification of orientations under parallel transport, we obtain:

$$|W^u(y_1) \times_{\mathrm{ev}_0} W^u(y_2)| \otimes \mathrm{ev}_0^* |M|_{y_1} \cong o_{y_1} \otimes o_{y_2}. \tag{4.4.10}$$

Since by assumption $\operatorname{ind}(y_0) = \operatorname{ind}(y_1) + \operatorname{ind}(y_2) - n$, the second transversality condition yields the isomorphism:

$$T(W^u(y_1) \times_{\mathrm{ev}_0} W^u(y_2)) \oplus TW^s(y_0) \xrightarrow{\sim} T\mathcal{L}^{r_1+r_2}M,$$

which on the level of orientation lines induces the isomorphism:

$$|\mathcal{L}^{r_1+r_2}M| \cong |W^u(y_1) \times_{\mathrm{ev}_0} W^u(y_2)| \otimes |W^s(y_0)|.$$

One final isomorphism that we need is the one from Lemma 4.3.3:

$$|\mathcal{L}^{r_1+r_2}M| \cong o_{y_0} \otimes |W^s(y_0)|$$

giving us the identification

$$|W^u(y_1) \times_M W^u(y_2)| \cong o_{y_0}. \tag{4.4.11}$$

Combining (4.4.10) and (4.4.11), we arrive at Equation (4.4.8) which we showed to be equivalent to the claim of the lemma. □

We are now in the position to define the product on the chain level:

$$\begin{aligned} * : CM_i(f^r; \mathrm{ev}_0^* |M|^{-1}) \otimes CM_j(f^{r_2}; \mathrm{ev}_0^* |M|^{-1}) &\to CM_{i+j}(f^{r_1+r_2}; \mathrm{ev}_0 |M|^{-1}) \\ *|(o_{y_1} \otimes \mathrm{ev}_0^* |M|_{y_1}^{-1}) \otimes (o_{y_2} \otimes \mathrm{ev}_0^* |M|_{y_2}^{-1}) &:= \sum_{\substack{\operatorname{ind}(y_0)=\operatorname{ind}(y_1)+\operatorname{ind}(y_2)-n \\ \gamma \in \mathcal{T}(y_0;y_1,y_2)}} *_\gamma. \end{aligned} \tag{4.4.12}$$

This product $*$ preserves the degree since $o_y \otimes |M|^{-1}$ is supported in degree

$$\operatorname{ind}(y) - n.$$

The next step is to prove that the differential satisfies the Leibniz rule with respect to the product.

Lemma 4.4.10 ([Abo14, Lem. 3.4.17.]). *Given $a_i \in o_{y_i} \otimes \mathrm{ev}_0^* |M|_{y_i}^{-1}$ for $i \in \{1, 2\}$, we have*

$$\partial(a_1 * a_2) = \partial a_1 * a_2 + (-1)^{\deg(a_1)} a_1 * \partial a_2. \tag{4.4.13}$$

Proof (Sketch). Whenever $\mathcal{T}(y_0; y_1, y_2)$ has dimension 1, it admits a natural compactification with boundary strata

$$\begin{aligned} &\bigsqcup_{\operatorname{ind}(y_0)=\operatorname{ind}(y_0')-1} \mathcal{T}(y_0; y_0') \times \mathcal{T}(y_0'; y_1, y_2) \\ &\bigsqcup_{\operatorname{ind}(y_1')=\operatorname{ind}(y_1)-1} \mathcal{T}(y_0; y_1', y_2) \times \mathcal{T}(y_1'; y_1) \\ &\bigsqcup_{\operatorname{ind}(y_2')=\operatorname{ind}(y_2)-1} \mathcal{T}(y_0; y_1, y_2') \times \mathcal{T}(y_2'; y_2). \end{aligned} \tag{4.4.14}$$

The elements of the first set correspond to $\partial(a_1 * a_2)$ and the other two to $\partial a_1 * a_2$ and $a_1 * \partial a_2$. In order to determine the sign, we have to compare the natural

orientations that arise at the boundary of the moduli space with the orientation stemming from the product.

Given an element of $\mathcal{T}(y_0; y_1, y_2)$, we will only analyse the stratum (4.4.14). The other strata follow in an analogous manner. Thus given an element of this boundary stratum, let γ_2 denote the flow line in $\mathcal{T}(y_2'; y_2)$. The vector $-\frac{d\gamma_2}{ds}$ corresponds to an outward pointing normal vector field at the boundary of $W^u(y_2)$. This yields an isomorphism

$$|\mathbb{R} \cdot \partial_s|^{-1} \otimes |W^u(y_1) \times_M W^u(y_2')| \cong |W^u(y_1) \times_{\mathrm{ev}_0} W^u(y_2)|.$$

Applying Equation (4.4.9), we obtain

$$|\mathbb{R} \cdot \partial_s|^{-1} \otimes |W^u(y_1) \times_M W^u(y_2')| \otimes \mathrm{ev}_0^* |M|_{y_1} \cong o_{y_1} \otimes o_{y_2}.$$

Using (4.4.11) and introducing a Koszul sign of $\mathrm{ind}(y_1)$ we have the following chain of isomorphisms

$$\begin{aligned}
&|\mathbb{R} \cdot \partial_s|^{-1} \otimes o_{y_0} \otimes \mathrm{ev}_0^* |M|_{y_1} \cong o_{y_1} \otimes o_{y_2}, \\
&o_{y_0} \otimes \mathrm{ev}_0^* |M|_{y_1} \cong |\mathbb{R} \cdot \partial_s|^{-1} \otimes o_{y_1} \otimes o_{y_2}, \\
&o_{y_0} \otimes \mathrm{ev}_0^* |M|_{y_1} \cong o_{y_1} \otimes |\mathbb{R} \cdot \partial_s|^{-1} \otimes o_{y_2} \cong o_{y_1} \otimes o_{y_2'},
\end{aligned}$$

where we used the isomorphism $\mathbb{R} \cdot \partial_s|^{-1} \otimes o_{y_2} \cong o_{y_2'}$, which defines the differential. In order to link this last isomorphism to the product, we must reverse the steps in Equation (4.4.6)-(4.4.8). This introduces a Koszul sign of $n(\mathrm{ind}(y_2) - 1)$ in the definition of $a_1 * \partial a_2$ and $n \,\mathrm{ind}(y_2)$ in the definition of $\partial(a_1 * a_2)$. Thus the total Koszul sign equals

$$n \,\mathrm{ind}(y_2) + n(\mathrm{ind}(y_2) - 1) + \mathrm{ind}(y_1) \equiv \mathrm{ind}(y_1) - n \mod 2$$

which is exactly the degree of the generator a_1 in Equation (4.4.13). □

Along the same lines, one shows that $*$ and ι commute, leading us to:

Corollary 4.4.11. *The product defined in (4.4.12) at the chain level descends to a map on homology that is compatible with the inclusion maps leading to a product map*

$$* : H_*(\mathcal{L}M; |M|^{-1}) \otimes H_*(\mathcal{L}M; |M|^{-1}) \to H_*(\mathcal{L}M; |M|^{-1}) \tag{4.4.15}$$

on loop homology with coefficients in $|M|^{-1}$.

4.4.c) Loop Product with Other Coefficients

So far, we have only constructed the loop product with coefficients in $\mathrm{ev}_0^* |M|^{-1}$. Under certain restrictions, we can extend this definition to other, twisted, coefficients.

By construction, an element $\mathcal{T}(y_0; y_1, y_2)$ corresponds to a triple of Morse trajectories

$$\begin{aligned}
\gamma_1 &: (-\infty, 0] \to \mathcal{L}^{r_1} M, \\
\gamma_2 &: (-\infty, 0] \to \mathcal{L}^{r_2} M, \\
\gamma_0 &: [0, +\infty) \to \mathcal{L}^{r_1 + r_2} M,
\end{aligned}$$

such that γ_i converges at infinity to y_i and the loop $\mathrm{geo}(\gamma_0(0))$ is obtained by concatenating $\mathrm{geo}(\gamma_1(0))$ and $\mathrm{geo}(\gamma_2(0))$.

Assume that we are given a local system ν on $\mathcal{L}M$ (of degree 0) with isomorphism

$$\nu_{\mathrm{geo}(\gamma_1(0))} \otimes \nu_{\mathrm{geo}(\gamma_2(0))} \to \nu_{\mathrm{geo}(\gamma_0(0))}. \tag{4.4.16}$$

Using the homotopy between $\mathrm{geo}(y_i)$ and $\mathrm{geo}(y_i(0))$, we obtain maps

$$\nu_\gamma : \nu_{\mathrm{geo}(y_1)} \otimes \nu_{\mathrm{geo}(y_2)} \to \nu_{\mathrm{geo}(y_0)}. \tag{4.4.17}$$

We can now extend the product:

$$\begin{aligned} &* : CM_i(f^{r_1}; \mathrm{ev}_0^* |M|^{-1} \otimes \nu) \otimes CM_j(f^{r_2}; \mathrm{ev}_0^* |M|^{-1} \otimes \nu) \to CM_{i+j}(f^{r_1+r_2}; \mathrm{ev}_0^* |M|^{-1} \otimes \nu) \\ &\quad *|(o_{y_1} \otimes \mathrm{ev}_0^* |M|_{y_1}^{-1} \otimes \nu_{y_1}) \otimes (o_{y_2} \otimes \mathrm{ev}_0^* |M|_{y_2}^{-1} \otimes \nu_{y_2}) := \sum_{\substack{\mathrm{ind}(y_0)=\mathrm{ind}(y_1)+\mathrm{ind}(y_2)-n \\ \gamma \in \mathcal{T}(y_0;y_1,y_2)}} *_\gamma \otimes \nu_\gamma. \end{aligned}$$

In order for the above product to be a chain map, we require that the isomorphism of Equation (4.4.16) is invariant under homotopies. Moreover, the product is associative if the isomorphism (4.4.16) is also associative.

LEMMA 4.4.12. *Both local systems* $\mathrm{ev}_0^*(|M|[n])^{-\otimes w(\gamma)}$ *and* σ *are equipped with isomorphisms as in (4.4.16).*

Proof. Note that

$$\mathrm{ev}_0^*(|M|[n])^{-\otimes w(\gamma)} \begin{cases} \mathrm{ev}_0^*(|M|[n]) & \text{if } w(\gamma) = -1 \\ \mathbb{Z} & \text{if } w(\gamma) = 0. \end{cases}$$

Recall that any two orientation on the number 8 induce an orientation on the remaining circle. Thus

$$w(\mathrm{geo}(\gamma_1)(0)) + w(\mathrm{geo}(\gamma_2)(0)) \equiv w(\mathrm{geo}(\gamma_0)(0)) \mod 2. \tag{4.4.18}$$

Since $\mathrm{ev}_0(\mathrm{geo}(\gamma_0(0)) = \mathrm{ev}_0(\mathrm{geo}(\gamma_1(0)) = \mathrm{ev}_0(\mathrm{geo}(\gamma_2(0))$, we have

$$\mathrm{ev}_0^*(|M|[n])_{\gamma_0(0)} \cong \mathrm{ev}_0^*(|M|[n])_{\gamma_1(0)} \cong \mathrm{ev}_0^*(|M|[n])_{\gamma_2(0)}. \tag{4.4.19}$$

Combining (4.4.18) and (4.4.19) yields the desired isomorphism in all cases.

As for the second local system, recall that σ can also be characterised by it being the local system of Spin structures. The claim now follows by noting that a Spin structure on a vector bundle over a pair of pants is determined by its restriction to any two of the three components of its boundary, just like in the case for orientation. Thus Spin structures over $\mathrm{geo}(\gamma_1(0))$ and $\mathrm{geo}(\gamma_2(0))$ fix a Spin structure over $\mathrm{geo}(\gamma_0(0))$. □

5 Going from Symplectic Cohomology to Loop Homology

In this section, we relate the symplectic cohomology of T^*M to the homology of the free loop space. We will do so by constructing a map

$$\mathcal{V}_r : CF^*(H;\mathbb{Z}) \to CM_{-*}(f^r;\eta)$$

for H a fixed linear Hamiltonian and r sufficiently large. The section will culminate in the proof of:

Theorem ([Abo14, Thm. 4.1.1.]). There is a map of rings from the symplectic cohomology of T^*M to the homology of the free loop space of M twisted by η, with reversed grading, i.e. we have a map

$$\mathcal{V} : SH^*(T^*M;\mathbb{Z}) \to H_{-*}(\mathcal{L}M;\eta)$$

preserving the operations e and $*$.

Remark 5.0.1. If ν is a local system on $\mathcal{L}M$, our method of proof similarly yields that the symplectic cohomology of T^*M with coefficients in ν is isomorphic to the homology of free loop space of M twisted by $\eta \otimes \nu$, up to a grading reversal. In particular, we have a map

$$SH^*(T^*M, \eta^{-1}) \to H_{-*}(\mathcal{L}M;\mathbb{Z}).$$

If ν is obtained by transgressing a local system on M, then the map we obtain will preserve the ring structure. We will follow [Abo14, Sec.4].

5.1 The Maslov Index for Loops

In what is to come, we will need to assign an index to loops (or paths) of Lagrangian subspaces in $\mathbb{C}^n$, as they crop up naturally in the analysis of linearisations of pseudoholomorphic equations with Lagrangian boundary conditions. We will start with a discussion of the topology of the Grassmannian of Lagrangians.

The subsequent lemma helps us understand the topology of the Grassmannian of Lagrangian subspaces of $\mathbb{C}^n$.

Lemma 5.1.1 ([Abo14, Lem. 4.2.1.]). *The Grassmannian* $\mathrm{Gr}(\mathbb{C}^n)$ *of Lagrangian subspaces in* $\mathbb{C}^n$ *is connected, and whenever* $3 \leq n$*, the fist two homotopy groups are*

$$\pi_1(\mathrm{Gr}(\mathbb{C}^n)) \cong \mathbb{Z}$$
$$\pi_2(\mathrm{Gr}(\mathbb{C}^n)) \cong \mathbb{Z}/2\mathbb{Z}.$$

Proof. In order to show this result, we are going to use two different fibrations. The group $U(n)$ acts transitively on the space of Lagrangians, with stabiliser homotopy equivalent to $O(n)$. This induces an isomorphism

$$\mathrm{Gr}(\mathbb{C}^n) \cong U(n)/O(n),$$

this is for example the content of [**McSal**]. We can thus analyse the long exact sequence in homotopy associated to the fibration $O(n) \to U(n) \to U(n)/O(n)$:

$$\ldots \longrightarrow \pi_2(U(n)) \longrightarrow \pi_2(U(n)/O(n)) \longrightarrow \pi_1(O(n)) \longrightarrow \pi_1(U(n)) \longrightarrow \ldots$$

For $n \geq 3$, we have that $\pi_2(U(n)) = 0$ and $\pi_1(O(n)) = \mathbb{Z}/2\mathbb{Z}$ in addition to the fact that the green map is the zero map. Thus $\pi_2(\mathrm{Gr}(\mathbb{C}^n)) \cong \mathbb{Z}/2\mathbb{Z}$.

In order to construct the second fibration, consider the homomorphism

$$d := \det^2 : U(n) \to S^1;$$

the square of the determinant. Multiplying by an element of $O(n)$ does not change the output of d and thus d factors as

$$\bar{d} : U(n)/O(n) \to S^1.$$

This map is in fact a fibration, with fibre equal to $SU(n)/SO(n)$. Since $SU(n)$ is simply connected and $SO(n)$ is connected, it follows from the long exact sequence in homotopy that $SU(n)/SO(n)$ is also simply-connected. Thus considering the relevant part of the long exact sequence of the fibration $\bar{d}$ we have:

$$\ldots \longrightarrow \underbrace{\pi_1(SU(n)/SO(n))}_{=0} \longrightarrow \pi_1(U(n)/O(n)) \xrightarrow{\pi_1(\bar{d})} \pi_1(S^1) \longrightarrow 0.$$

□

In terms of cohomology, we obtain an isomorphism

$$H^1(\mathrm{Gr}(\mathbb{C}^n), \mathbb{Z}) \cong \mathbb{Z}$$
$$H^2(\mathrm{Gr}(\mathbb{C}^n), \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$$

We will now present a more concrete construction of the cohomological isomorphism. Let I be the standard complex structure on $\mathbb{C}^n$. Observe that given a Lagrangian subspace L of $\mathbb{C}^n$, we have that

$$IL \cap L = \{0\}.$$

This implies that a real basis for L defines a complex basis for $\mathbb{C}^n$. Writing as before det for the top exterior power of a real vector space, and $\det_{\mathbb{C}}$ for the top complex exterior power, the inclusion of L in $\mathbb{C}^n$ induces an inclusion

$$\det_{\mathbb{R}}(L) \to \det_{\mathbb{C}}(\mathbb{C}^n) \tag{5.1.1}$$

which is canonical up to homotopy. Fixing an identification of $\det_{\mathbb{C}}(\mathbb{C}^n) \cong \mathbb{C}$ with the plane, we obtain a map

$$\mathrm{Gr}(\mathbb{C}^n) \to \mathbb{R}\mathrm{P}^1 = S^1 \tag{5.1.2}$$

which maps a Lagrangian subspace to the real line $\det_{\mathbb{R}}(L)$, seen as a line in $\det_{\mathbb{C}}(\mathbb{C}^n) \cong \mathbb{C}$ via Equation (5.1.1).

Definition 5.1.2. *The topological Maslov index μ is the class in $H^1(\mathrm{Gr}(\mathbb{C}^n), \mathbb{Z})$ associated to Equation (5.1.2).*

More concretely, the Maslov index assigns to a loop in $\mathrm{Gr}(\mathbb{C}^n)$ the winding number around S^1 of its image under (5.1.2).

We will also need a standard result from algebraic topology. Let $\Phi : S^1 \to U(n)$ be a loop. By a clutching construction, we can associate to Φ a unitary bundle E_Φ over $\mathbb{C}\mathrm{P}^1$. Indeed, by gluing the trivial bundle (z, e) on the unit disc centred at z_0 to the trivial bundle on its complement centred at z_∞ via the map

$$(e^{i\theta}, e) \mapsto (e^{i\theta}, \Phi_\theta^{-1}(e)).$$

By then choosing a connection, we obtain an operator:

$$D_\Phi^- : W^{1,p}(\mathbb{C}\mathrm{P}^1, E_\Phi) \to L^p(\mathbb{C}\mathrm{P}^1, E_\Phi \otimes \Omega^{0,1}\,\mathbb{C}\mathrm{P}^1),$$

which is a complex linear operator. Thus both the kernel and cokernel are complex vector spaces and hence have natural orientations as real vector spaces. We define $\det_{\mathbb{R}}(D_\Phi^-)$ to be the determinant line of D_Φ^- over the real numbers. The degree of this line is equal to the real index of D_Φ^-:

$$\mathrm{ind}_{\mathbb{R}}(D_\Phi^-) = \dim_{\mathbb{R}}(\ker(D_\Phi^-)) - \dim_{\mathbb{R}}(\mathrm{coker}(D_\Phi^-)).$$

We then have the result from [Hat02, Sec. 1.2.]:

Lemma 5.1.3 ([Abo14, Lem. 1.4.9.]). *The map*

$$\rho : \pi_1(U(n)) \to \mathbb{Z}$$
$$\rho(\Phi) = \frac{\mathrm{ind}_{\mathbb{R}}(D_\Phi^-)}{2} - n$$

is an isomorphism which agrees with the homomorphism

$$\Phi \mapsto \deg(\det \circ \Phi),$$

where $\det : U(n) \to S^1$ *is the determinant homomorphism.*

As it will be useful for later calculations, consider a loop Λ in $\mathrm{Gr}(\mathbb{C}^n)$ together with a loop Φ in $U(n)$. Define

$$\Phi(\Lambda)_t := \Phi_t(\Lambda_t).$$

By comparing the definition of the Maslov index with Lemma 5.1.3, we find that

$$\mu(\Phi_t(\Lambda_t)) = \mu(\Lambda) - 2\rho(\Phi).$$

More abstractly, this implies that the map

$$H_1(\mathrm{Sp}(2n, \mathbb{R}); \mathbb{Z}) \oplus H_1(\mathrm{Gr}(\mathbb{C}^n); \mathbb{Z}) \to H_1(\mathrm{Gr}(\mathbb{C}^n); \mathbb{Z})$$

corresponds under our chosen trivialisation of the first homology groups of $\mathrm{Gr}(\mathbb{C}^n)$ and $\mathrm{Sp}(2n, \mathbb{R})$ with the map

$$\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$$
$$(a, b) \mapsto b - 2a.$$

In order to identify the second cohomology group of $\mathrm{Gr}(\mathbb{C}^n)$, it suffices to give a non-zero element. For this consider the diagram

$$\begin{array}{ccccc} O(n) & \longrightarrow & U(n) & \longrightarrow & \mathrm{Gr}(\mathbb{C}^n) \\ \downarrow & & \downarrow & & \downarrow \\ O(n)\times O(n) & \longrightarrow & O(2n) & \longrightarrow & O(2n)/(O(n)\times O(n)). \end{array}$$

Using the naturality of the long exact sequence in homotopy of fibrations, we see that

$$\pi_2(\mathrm{Gr}(\mathbb{C}^n)) \cong O(2n)/(O(n)\times O(n));$$

and thus also

$$H^2(\mathrm{Gr}(\mathbb{C}^n);\mathbb{Z}) \cong H^2(O(2n)/(O(n)\times O(n));\mathbb{Z}).$$

By definition, the second Stiefel-Whitney class of the tautological bundle over $O(2n)/(O(n)\times O(n))$ does not vanish for $n>1$, and so

$$w_2(E) \in H^2(\mathrm{Gr}(\mathbb{C}^n);\mathbb{Z}/2\mathbb{Z}) \neq 0.$$

From these discussions, we can also draw conclusions about local systems on the free loop space $\mathcal{L}\,\mathrm{Gr}(\mathbb{C}^n)$. Recall that there is a natural transgression map

$$H^k(X;\mathbb{Z}) \to H^{k-1}(\mathcal{L}X;\mathbb{Z})$$

from any space X, which is dual to a map which assigns to any cycle in the loop space the cycle in the base swept by the corresponding family of loops. We also have a natural map

$$\mathrm{ev}_0 : H^k(X;\mathbb{Z}) \to H^k(\mathcal{L}X;\mathbb{Z})$$

induced by the evaluation at zero map.

For $k=0$, the fact that the fundamental group of $\mathrm{Gr}(\mathbb{C}^n)$ is abelian implies that the components of $\mathcal{L}\,\mathrm{Gr}(\mathbb{C}^n)$ are in bijective correspondence with elements of the fundamental group. This implies that the transgression map

$$\mathbb{Z} \cong H^1(\mathrm{Gr}(\mathbb{C}^n);\mathbb{Z}) \to H^0(\mathcal{L}\,\mathrm{Gr}(\mathbb{C}^n);\mathbb{Z})$$

is an isomorphism. The free loop space $\mathcal{L}\,\mathrm{Gr}(\mathbb{C}^n)$ thus presents a grading by the Maslov index, and we will denote by $\mathcal{L}^\mu\,\mathrm{Gr}(\mathbb{C}^n)$ the components corresponding to loops of index μ.

Proposition 5.1.4 ([Abo14, Prop. 4.2.5.]). *For $n>2$, transgression and evaluation maps induce the isomorphism*

$$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} \cong H^1(\mathrm{Gr}(\mathbb{C}^n);\mathbb{Z})\oplus H^2(\mathrm{Gr}(\mathbb{C}^n);\mathbb{Z}) \to H^1(\mathcal{L}\,\mathrm{Gr}(\mathbb{C}^n);\mathbb{Z}). \tag{5.1.3}$$

Proof (Sketch). From Bott periodicity, it follows that $\mathrm{Gr}(\mathbb{C}^n)$ is an H-space. Since the free loop space of an H-space splits as the product of the based loop space, we conclude by the Künneth formula that

$$H^1(\mathcal{L}\,\mathrm{Gr}(\mathbb{C}^n);\mathbb{Z}) \cong H^1(\Omega\,\mathrm{Gr}(\mathbb{C}^n);\mathbb{Z})\oplus H^1(\mathrm{Gr}(\mathbb{C}^n);\mathbb{Z}).$$

When $n>2$, we have that $\pi_1(\Omega\,\mathrm{Gr}(\mathbb{C}^n)) \cong \pi_2(\mathrm{Gr}(\mathbb{C}^n)) \cong \mathbb{Z}/2\mathbb{Z}$, and thus we are able to conclude. □

This result has the following consequences for local systems on $\mathcal{L}\,\mathrm{Gr}(\mathbb{C}^n)$): denote by $\mathrm{ev}_0^*\,|E|$ the pullback under the evaluation map of the local system of orientations of the tautological bundle, and by σ^E the transgression of E to a local system on the free loop space. Reducing Equation (5.1.3) modulo 2, we find

$$H^1(\mathcal{L}\,\mathrm{Gr}(\mathbb{C}^n);\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

In the correspondence between elements of this group and rank-1 local systems, $\mathrm{ev}_0^*\,|E|$ and σ^E give rise to distinct (non-vanishing) elements. We have:

Corollary 5.1.5 ([Abo14, Cor. 4.2.6]). *Every non-trivial local system over $\mathcal{L}\,\mathrm{Gr}(\mathbb{C}^n)$ is isomorphic to either $\mathrm{ev}_0^*\,|E|$, σ^E or their tensor product.*

5.2 Construction of the Chain Map

At last, we are in a position to relate both Floer complex of a Hamiltonian, that we defined in Section 2, and the Morse chains with coefficients in η of finite dimensional approximations of the free loop space of Section 4. Eventually, we will show that the map that we will subsequently define between these two objects in addition commutes with continuation maps, thus we obtain a map from symplectic cohomology to the homology of the free loop space with coefficients in η.

5.2.a) Punctured Discs with Boundary on the Zero Section

Consider the punctured disc $D^2 \setminus \{(0,0)\}$ which we identify with the half-cylinder

$$Z^+ := [0,+\infty) \times S^1.$$

Given a linear Hamiltonian H whose periodic orbits are non-degenerate, we choose a family $H^+_{s,t}$ of Hamiltonians of equal slope, with Hamiltonian flow $X^+_{s,t}$ such that

$$X^+_{0,t}|_M \equiv 0 \quad \text{and} \quad H^+_{s,t} = H_t \quad \text{whenever } s \gg 0. \tag{5.2.1}$$

Similarly, we extend J_t to a family $J^+_{s,t}$ parametrised by points in Z^+ and define $\mathcal{M}(x)$, for each orbit $x \in \mathcal{O}(H)$, to be the moduli space of maps

$$u : Z^+ \to T^*M$$

solving the differential equation

$$(Du - dt \otimes X^+_{s,t})^{0,1} = 0, \tag{5.2.2}$$

with asymptotic condition x at $s = +\infty$, and with boundary conditions

$$u(0,t) \in M.$$

By associating to each map u the loop $u(0,t)$, we obtain an evaluation map

$$\mathrm{ev} : \mathcal{M}(x) \to \mathcal{L}M.$$

In order to understand the tangent space of $\mathcal{M}(x)$, we must compute its virtual dimension. Recall that $\deg(x)$ is defined in terms of an operator D_{ψ_x} constructed

from $x^*(TT^*M)$ with respect to a fixed trivialisation: under this pullback, the tangent space of M maps to a constant loop if $x^*(T^*M)$ is orientable, and to a loop of Maslov index 1 otherwise.[8]

This preferred trivialisation extends to the punctured disc, so we can also use it to linearise the operator associated to Equation (5.2.2) to obtain an operator

$$D_u : W^{1,p}((Z^+, S^1), (\mathbb{C}^n, \Lambda_x)) \to L^p(Z^+, \mathbb{C}^n),$$

where the notation indicates that we take $W^{1,p}$ maps from Z^+ to $\mathbb{C}^n$ with boundary conditions at $s = 0$ mapping to the loop Λ_x of Lagrangians in $\mathbb{C}^n$, which is the image of TM under the trivialisation.

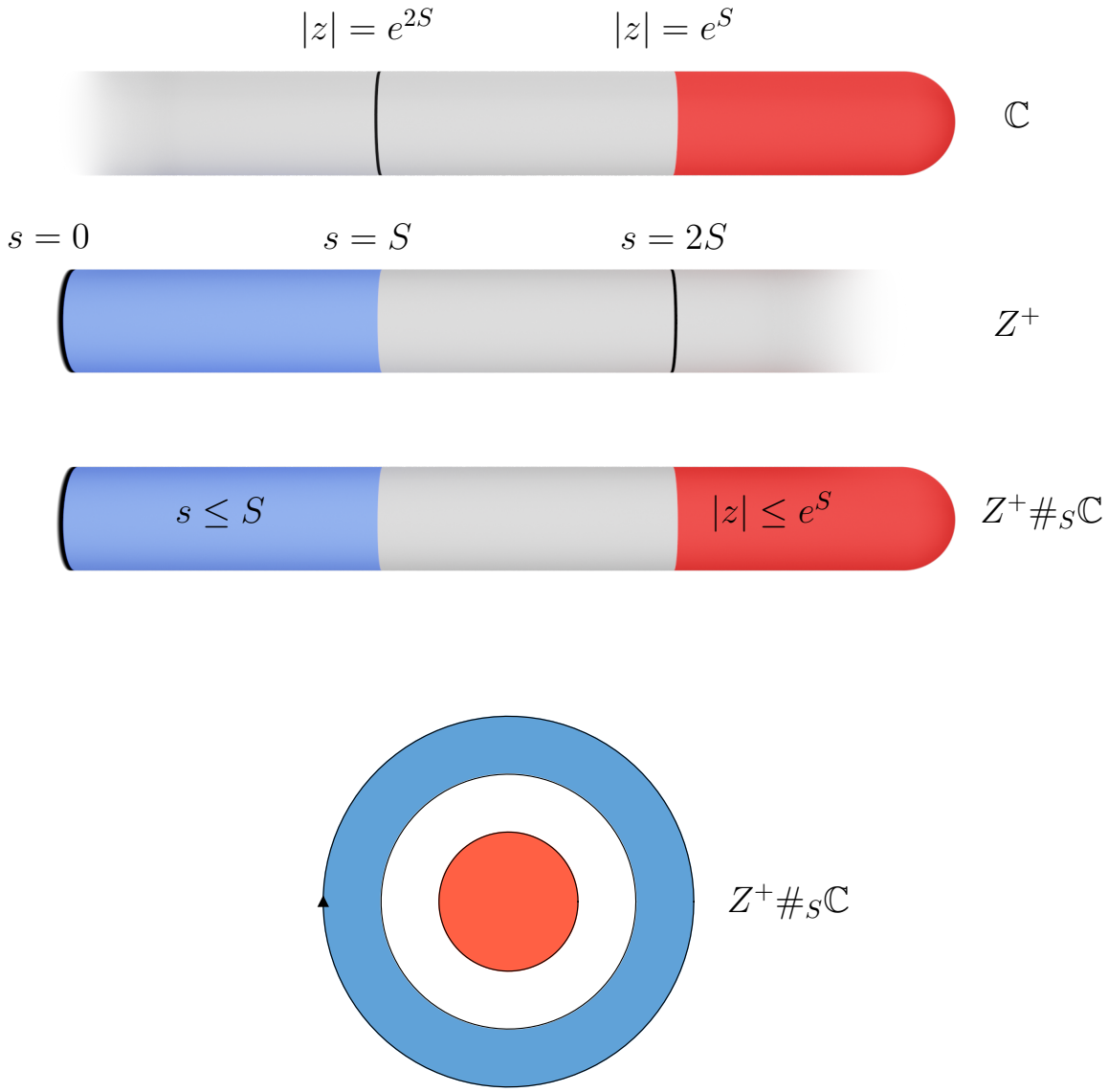


Figure I.5: Gluing for the chain map

[8]Note that this will turn out to be the reason for the introduction of the shift of Equation (2.4.9).

Since $D_{\psi_x} : W^{1,p}(\mathbb{C}, \mathbb{C}^n) \to L^p(\mathbb{C}, \mathbb{C}^n)$, we can glue D_u to D_{ψ_x} along their common end: the result is an operator $D_u \# D_{\psi_x}$ on a disc with no puncture, and with Lagrangian boundary conditions Λ_x going clockwise in Figure I.5 . Since $\pi \circ u$ defines a homotopy between $\pi \circ x$ and $\mathrm{ev}(u)$, we obtain an isomorphism of graded lines

$$\det(T_u \mathcal{M}(x)) \otimes \det(D_{\psi_x}) \cong \det(D_{\Lambda^{-1}_{\mathrm{ev}(u)}}), \tag{5.2.3}$$

where $D_{\Lambda^{-1}_{\mathrm{ev}(u)}}$ is the Cauchy-Riemann operator on the disc with Lagrangian boundary conditions obtained by applying the trivialisation Φ to TM along $\mathrm{ev}(u)$. We have the following result:

Lemma 5.2.1 ([Abo14, Lem. 4.3.1.]). *The index of $D_{\Lambda^{-1}_{\mathrm{ev}(u)}}$ is $n + w(x)$. Moreover, there is a canonical isomorphism*

$$|\det(D_{\Lambda^{-1}_{\mathrm{ev}(u)}})| \cong \eta_x[-w(x)]. \tag{5.2.4}$$

$\square_{\text{ref.}}$

This has the following important consequence:

Lemma 5.2.2 ([Abo14, Lem. 4.3.2.]). *The virtual dimension of the moduli space $\mathcal{M}(x)$ is equal to $n - \deg(x)$. Moreover, there is a natural isomorphism*

$$|\mathcal{M}(x)| \otimes o_x[w(x)] \cong \eta_x. \tag{5.2.5}$$

Proof. The isomorphism of lines follows from Equation (5.2.3) and (5.2.4). To compute the virtual dimension, we use the additivity of the index and Lemma 5.2.1 to obtain:

$$\mathrm{ind}(D_u) = n + w(x) - \mathrm{ind}(D_{\psi_x}).$$

Since $\deg(x) = \mathrm{ind}(D_{\psi_x}) - w(x)$, we conclude that

$$\mathrm{ind}(D_u) = n - \deg(x),$$

which gives the desired formula for the expected dimension. $\square$

5.2.b) Structure of the Compactified Moduli Space

The Gromov-Floer compactification $\overline{\mathcal{M}}(x)$ of $\mathcal{M}(x)$ is obtained by adding the strata:

$$\bigsqcup_{\substack{k \\ x_0, \dots, x_{k-1}}} \mathcal{M}(x_0) \times \mathcal{M}(x_0; x_1) \times \cdots \times \mathcal{M}(x_{k-1}; x).$$

The integer k will be called the *virtual codimension* of the stratum. From now on, we will pose that

(H2) all elements of the moduli spaces $\mathcal{M}(x_0)$ and $\mathcal{M}(x_i, x_{i+1})$ are regular, and have image contained in D^*M.

The fact that $\overline{\mathcal{M}}(x)$ is compact is as always guaranteed by the fact that the image is contained in D^*M.

In contrary to what we considered up until now, we will consider moduli spaces of arbitrary dimension. Concretely, given r marked points $(z_1, \dots, z_r)$ on the boundary of the punctured disc, we shall consider the evaluation map

$$\begin{aligned} \overline{\mathcal{M}}(x) &\to M^r \\ v &\mapsto (v(z_1), \dots, v(z_r)). \end{aligned}$$

In what is to follow, we will sometimes need to consider proper maps

$$\iota : N \to M^r$$

from a manifold with boundary N to M^r. Whenever the codimension of N equals $n - \deg(x)$, Sard's theorem implies that, for an open dense subset of maps ι, the inclusion

$$\mathcal{M}(x) \times_{M^r} N \subset \overline{\mathcal{M}}(x) \times_{M^r} N$$

is an equality, and that the left hand side is a compact 0-dimensional submanifold obtained as a transverse fibre product. Indeed, a generic perturbation of ι ensures that the image of ι is disjoint from all the boundary strata of $\overline{\mathcal{M}}(x)$. Because ι is a proper map, we see that the inverse image of N is contained in a compact subset of $\mathcal{M}(x)$. A standard application of Sard's theorem therefore implies that the fibre product is generically transverse.

We also need the case where the virtual dimension of the fibre product is one:

Proposition 5.2.3 ([Abo14, Prop. 4.3.3.]). *Assume that Hypothesis (H2) holds, and that the codimension of N is $n + 1 - deg(x)$. After a generic perturbation of ι, $\overline{M}(x) \times_{M^r} N$ is a compact manifold of dimension 1, whose boundary can be naturally decomposed into the strata*

$$\bigsqcup_{\deg(x_0)=\deg(x)+1} (\mathcal{M}(x_0) \times_{M^r} N) \times \mathcal{M}(x_0, x), \quad \text{and}$$
$$\overline{\mathcal{M}}(x) \times_{M^r} \partial N.$$

$\square_{\text{ref.}}$

5.2.c) Construction of the Map on Floer Cohomology

We are now at leisure to link the Floer cochain complex to the Morse complex of a finite dimensional approximation for r large enough. The starting point is the map

$$\begin{aligned} \mathrm{ev}_r : \overline{\mathcal{M}}(x) &\to M^r \\ u &\mapsto \Big(\mathrm{ev}(u)(0), \dots, \mathrm{ev}(u)(\frac{i}{r}), \dots, \mathrm{ev}(u)\Big(\frac{r-1}{r}\Big)\Big). \end{aligned} \tag{5.2.6}$$

Lemma 5.2.4 ([Abo14, Lem. 4.3.5.]). *If r is large enough, the image of ev_r lies in $\mathcal{L}^r M$. Moreover, the following diagram commutes up to homotopy:*

$$\begin{array}{ccc} \overline{\mathcal{M}}(x) & \xrightarrow{\mathrm{ev}_r} & \mathcal{L}^r M \\ {\scriptstyle \mathrm{ev}}\downarrow & \searrow^{\mathrm{ev}_{r+1}} & \downarrow{\scriptstyle \iota} \\ \mathcal{L}M & \xleftarrow[\mathrm{geo}]{} & \mathcal{L}^{r+1} M. \end{array}$$

Proof. The moduli space $\overline{\mathcal{M}}(x)$ is compact, thus there is a unique bound on the C^1-norm of the curves $\mathrm{ev}(u)$ for $u \in \overline{\mathcal{M}}(x)$. For each constant δ, we may therefore choose r large enough such that the restriction of $\mathrm{ev}(u)$ to any interval of length $\frac{1}{r}$ is less than δ. The image of ev_r hence lies in $\mathcal{L}^r M$ by (4.2.1).

This argument moreover shows that r can be chosen such that the image of ev_r between the points $\frac{i}{r}$ and $\frac{i+1}{r}$ is contained within the ball of radius 1 of either endpoint, for each i. The key observation is now that each such segment is homotopic within such a ball to the local geodesic connecting the endpoints; this argument hinges entirely on our standing assumption (H1) that the injectivity radius is larger than 4. There is a contractible choice of such local homotopies, which implies that we can choose them smoothly over each stratum of $\overline{\mathcal{M}}(x)$, and continuously over the whole space. Thus $\mathrm{geo} \circ \mathrm{ev}_{r+1}$ is homotopic to ev. The same argument produces a homotopy between ev_{r+1} and $\iota \circ \mathrm{ev}_r$. □

We now construct a sort of 'chimera' moduli space. Given a Hamiltonian orbit x together with a critical point y of f^r for r large enough to satisfy the conclusion of Lemma 5.2.4, we define

$$\mathcal{B}(y, x) := \mathcal{M}(x) \times_{\mathrm{ev}_r} W^s(y). \tag{5.2.7}$$

One can think of this space as consisting of punctured discs that converge to x at the puncture and whose points at the boundary flow towards y under f^r.

Lemma 5.2.5 ([Abo14, Lem. 4.3.7.]). *For a generic function f^r, the moduli space $\mathcal{B}(y, x)$ is a manifold of dimension*

$$n - \deg(x) - \mathrm{ind}(y). \tag{5.2.8}$$

Moreover, every element of $\mathcal{B}(y, x)$ determines a canonical isomorphism

$$o_y \otimes o_x[w(x)] \cong |\mathcal{B}(y, x)|^{-1} \otimes \eta_x. \tag{5.2.9}$$

Proof. The dimension of $\mathcal{M}(x)$ is $n - \deg(x)$, and the codimension of $W^s(y)$ is $\mathrm{ind}(y)$, thus the fibre product has the desired dimension. To check the statement about orientation lines, we start from the short exact sequence of the fibre product

$$T\mathcal{B}(y, x) \longrightarrow T\mathcal{M}(x) \oplus TW^s(y) \longrightarrow T\mathcal{L}^r M$$

which yields the chain of isomorphisms

$$\begin{aligned} |\mathcal{B}(y, x)| \otimes |\mathcal{L}^r M| &\cong |\mathcal{M}(x)| \otimes |W^s(y)|, \\ |\mathcal{L}^r M| &\cong |\mathcal{B}(y, x)|^{-1} \otimes |\mathcal{M}(x)| \otimes |W^s(y)|, \\ o_y \otimes |W^s(y)| &\cong |\mathcal{B}(y, x)|^{-1} \otimes |\mathcal{M}(x)| \otimes |W^s(y)|, \end{aligned}$$

where the last line used Equation (4.3.1). Simplifying and tensoring both sides with $o_x[w(x)]$ yields the isomorphism

$$o_y \otimes o_x[w(x)] \cong |\mathcal{B}(y, x)|^{-1} \otimes |\mathcal{M}(x)| \otimes o_x[w(x)].$$

Using Equation (5.2.5), we are able to conclude. □

Let us now consider the situation where $\mathcal{B}(y,x)$ is 0-dimensional. Using the natural identification of o_y with its inverse, we obtain from Equation (5.2.9) an isomorphism

$$\mathcal{V}_u : o_x[w(x)] \cong o_y \otimes \eta_y. \tag{5.2.10}$$

Next, since we previously fixed the Hamiltonian H, we choose r large enough so that Lemma 5.2.4 applies to all time-1 periodic orbits of H. We set

$$\begin{aligned} \mathcal{V}_r : CF^*(H;\mathbb{Z}) &\to CM_{-*}(f^r;\eta) \\ \mathcal{V}_r|_{o_x[w(x)]} &:= (-1)^{\deg(x)} \sum_{\substack{n-\mathrm{ind}(y)=\deg(x) \\ u\in\mathcal{B}(y,x)}} \mathcal{V}_u. \end{aligned} \tag{5.2.11}$$

In order to show that this map descends to cohomology, we need to show that it is a chain map. For this, consider a moduli space $\mathcal{B}(y,x)$ that is 1-dimensional. The natural compactification of this space is a manifold $\overline{\mathcal{B}}(y,x)$ with boundary

$$\bigsqcup_{\deg(x_0)=\deg(x_1)+1} \mathcal{B}(y,x_0)\times\mathcal{M}(x_0,x), \tag{5.2.12}$$

$$\bigsqcup_{\mathrm{ind}(y)=\deg(y_1)+1} \mathcal{T}(y,y_1)\times\mathcal{B}(y_1,x). \tag{5.2.13}$$

Lemma 5.2.6 ([Abo14, Lem. 4.3.8.]). *The map $\mathcal{V}_r$ is a chain map, i.e.*

$$\partial\circ\mathcal{V}_r = \mathcal{V}_r\circ d.$$

Proof. This proof will follow along the same lines as similar proof of previous sections. We will consider the two boundary strata of (5.2.12) and (5.2.13) separately, and then conclude by the fact that the orientations at the boundary of a 1-dimensional manifold must be of opposite sign.

First, consider a stratum of the type $\mathcal{B}(y,x_0)\times\mathcal{M}(x_0,x)$. The element $\mathcal{B}(y,x_0)$ induces an isomorphism

$$o_{x_0}[w(x_0)] \cong o_y \otimes \eta_y.$$

Given $u\in\mathcal{M}(x_0,x)$, the map ∂_u is induced by the natural isomorphism

$$o_{x_0}[w(x_0)] \cong |\partial_s u| \otimes o_x[w(x)]$$

coming from gluing. Combining the two isomorphisms yields an isomorphism

$$|\partial_s u| \otimes o_x[w(x)] \cong o_y \otimes \eta_y$$

connecting the orientation lines of x and y. We now note that the vector field $\partial_s u$ corresponds to an outward pointing vector along the boundary of $\overline{\mathcal{B}}(y,x)$. We can therefore identify $|\partial_s u|^{-1}$ and $|\mathcal{B}(y,x)|^{-1}$ in order to compare the map with Equation (5.2.9), which is the natural map on orientation lines induced by the definition of $\overline{\mathcal{B}}(y,x)$ as a fibre product. Explicitly, this comparison is done via the

following chain of isomorphisms, where we keep track of the total Koszul sign on the right:

$$o_y \otimes o_x[w(x)] \cong |\partial_s u|^{-1} \otimes \eta_y \qquad 0, \tag{5.2.14}$$
$$|\partial_s u| \otimes o_y \otimes o_x[w(x)] \cong \eta_y \qquad 1, \tag{5.2.15}$$
$$|\partial_s u| \otimes o_x[w(x)] \cong o_y^{-1} \otimes \eta_y \qquad \mathrm{ind}(y), \tag{5.2.16}$$
$$|\partial_s u| \otimes o_x[w(x)] \cong o_y \otimes \eta_y \qquad \mathrm{ind}(y). \tag{5.2.17}$$

Taking into account the sign of Equation (5.2.11), the natural map induced by $\partial\overline{\mathcal{B}}(y,x)$ along the stratum (5.2.12) agrees with $(-1)^{n+1}\mathcal{V}_r \circ d$ since in $\mathbb{Z}_2$, we have

$$\mathrm{ind}(y) + \deg(x_0) + n + 1 = n - \mathrm{ind}(y) + \deg(x_0) + 1 \qquad \mod 2, \tag{5.2.18}$$
$$= 2\deg(x_0) + 1 \qquad \mod 2, \tag{5.2.19}$$
$$= 1 \qquad \mod 2. \tag{5.2.20}$$

Next, we consider the boundary component associated to (5.2.13): given $\gamma \in \mathcal{T}(y, y_1)$, the map ∂_γ is defined by the isomorphism

$$|\mathbb{R} \cdot \partial_s \gamma|^{-1} \otimes o_{y_1} \cong o_y \qquad 0, \tag{5.2.21}$$
$$o_{y_1} \cong |\mathbb{R} \cdot \partial_s \gamma| \otimes o_y \qquad 1, \tag{5.2.22}$$
$$o_{y_1} \cong |\overline{\mathcal{B}}(y,x)| \otimes o_y \qquad 0. \tag{5.2.23}$$

The sign in the middle arises from the fact that we defined l^{-1} to be the **left** inverse to l in Equation 2.4.1, while the last sign comes from the fact that the vector field $-\partial_s\gamma$ corresponds to an inward pointing vector along the boundary of $\overline{\mathcal{B}}(y,x)$. Using the isomorphism obtained by applying Equation (5.2.9) to $\mathcal{B}(y_1,x)$, we arrive at the isomorphism for $\mathcal{B}(y,x)$.

The remaining sign arises from Lemma 1.7.1. 2.4.3, applied to $o_y \cong |\mathbb{R}\cdot\partial_s\gamma|^{-1} \otimes o_{y_1}$. The parity sign is

$$\mathrm{ind}(y_1).$$

Since $\mathrm{ind}(y_1) = \deg(x) + n \mod 2$, we conclude that the natural map induced by $\partial\overline{\mathcal{B}}(y,x)$ along this stratum agrees with $(-1)^n\partial \circ \mathcal{V}_r$. As the sum of the maps associated to $\partial\overline{B}(y,x)$ vanishes, we conclude that

$$(-1)^{n+1}\mathcal{V}_r \circ d + (-1)^n \partial \circ \mathcal{V}_r = 0.$$

□

5.3 Compatibility of $\mathcal{V}_r$ with the Defined Structures

In this section, we will justify why our freshly defined map descends to a map

$$\mathcal{V} : SH^*(T^*M;\mathbb{Z}) \to H_{-*}(\mathcal{L}M;\eta).$$

Furthermore, we will allude to the fact that $\mathcal{V}$ is compatible with all the operations that we defined on symplectic cohomology.

5.3.a) Compatibility with the Inclusion Maps

Assuming that the maps $\mathcal{V}_{r-1}$ and $\mathcal{V}_r$ are both defined, we claim that the diagram

$$\begin{array}{ccc} HF^*(H;\mathbb{Z}) & \xrightarrow{\mathcal{V}_{r-1}} & HM_{-*}(\mathcal{L}^{r-1}M;\eta) \\ & \searrow^{\mathcal{V}_r} & \downarrow \\ & & HM_{-*}(\mathcal{L}^{r}M;\eta) \end{array}$$

commutes. The proof will follow, as so often for these types of reasonings, from a cobordism argument. Assuming that $\mathrm{ind}(y) = n - \deg(x)$, consider, for generic data, the 1-dimensional fibre product

$$\mathcal{B}^\iota(y,x) := \bigcup_{t=0}^{\infty} \mathcal{M}(x) \times_{\iota \times \psi_t^r \circ \mathrm{ev}_{r-1}} W^s(y).$$

This manifold consists of arbitrary flow lines of $-\mathrm{grad}(f^r)$ emanating from $\mathcal{M}(x)$, whose image under ι intersects the stable manifold of y. By considering the boundaries $t = 0$ and $t = +\infty$, we see that this is a cobordism between

$$\mathcal{M}(x) \times_{\iota \circ \mathrm{ev}_{r-1}} W^s(y) \quad \text{, and} \tag{5.3.1}$$

$$\bigsqcup_{y'} \mathcal{B}(y',x) \times W^u(y') \times_\iota W^s(y). \tag{5.3.2}$$

Since trajectories can break at both ends, we have that $\mathcal{B}^\iota(y,x)$ admits a natural compactification to a 1-dimensional manifolds with boundary given by adding

$$\bigsqcup_{x_0} \mathcal{B}^\iota(y,x_0) \times \mathcal{M}(x_0,x) \quad \text{and} \quad \bigsqcup_{y_1} \mathcal{T}(y,y_1) \times \mathcal{B}^\iota(y_1,x). \tag{5.3.3}$$

Note that (5.3.2) corresponds to the composition $\iota \circ \mathcal{V}_{r-1}$. At the same time, we have that $\iota \circ \mathrm{ev}_{r-1}$ is homotopic to ev_r by Lemma 5.2.4 and thus we see that (5.3.1) corresponds to $\mathcal{V}_r$. Equation (5.3.3) then implies that $\mathcal{V}_r$ is homotopic to $\iota \circ \mathcal{V}_{r-1}$.

Lemma 5.3.1 ([Abo14, Lem. 4.3.10.]). *If r is sufficiently large, the composition*

$$HF^*(H;\mathbb{Z}) \longrightarrow HM_{-*}(\mathcal{L}^r M;\eta) \longrightarrow H_{-*}(\mathcal{L}M;\eta) \tag{5.3.4}$$

is independent of r.

Furthermore, an argument similar to the proof of Lemma 2.6.9 shows that the composition (5.3.4) is independent of the choice of Floer data.

5.3.b) Compatibility with the Continuation Maps

The proof that $\mathcal{V}_r$ commutes with continuation maps, i.e. that the diagram

$$\begin{array}{ccc} HF^*(H;\mathbb{Z}) & \longrightarrow & HF^*(K;\mathbb{Z}) \\ {\scriptstyle \mathcal{V}_r}\downarrow & \swarrow_{\mathcal{V}_r} & \\ HM_{-*}(\mathcal{L}^r M;\eta) & & \end{array}$$

commutes for sufficiently large r follows in a very similar way to the one of the inclusion maps and thus we refer to [Abo14, Sec. 4.3.5.]. We conclude

PROPOSITION 5.3.2 ([Abo14, Prop. 4.3.14.]). *The maps $\mathcal{V}_r$ induce a map*

$$\mathcal{V} : SH^*(T^*M;\mathbb{Z}) \to H_{-*}(\mathcal{L}M;\eta)$$

that is independent of all choices.

5.3.c) Compatibility with the Ring Operations

Since the map $\mathcal{V}$ gets induced by $\mathcal{V}_r$ through the colimit construction, we will fix, for each argument, a sufficiently large r and show that $\mathcal{V}_r$ commutes with the operation defined in Floer and Morse theory.

Recall that we defined the unit of Floer cohomology by counting elements of the moduli spaces $\mathcal{P}(x)$, whenever x was a time-1 orbit of a Hamiltonian H satisfying $\deg(x) = 0$. In a next step, we glue these rigid pseudo-holomorphic planes in T^*M to rigid pseudo-holomorphic discs of $\mathcal{M}(x)$. Note that the moduli space

$$\bigsqcup_x \mathcal{M}(x) \times \mathcal{P}(x)$$

consists of *broken* pseudo-holomorphic discs with boundary condition along the zero section. Gluing elements of this moduli space, we obtain an equation on the disc D^2, which is deformable into an equation without an inhomogeneous term, i.e. an equation given by

$$J \cdot \frac{\partial u}{\partial s} = \frac{\partial u}{\partial t},$$

with respect to some complex structure J on T^*M. We thus obtain a family of equations on the disc parametrised by an interval, and define $\mathcal{R}(M)$ to be the moduli space of solutions with boundary on the zero section.

LEMMA 5.3.3 ([Abo14, Lem. 4.4.2.]). *For generic choices of Cauchy-Riemann equations, the space $\mathcal{R}(M)$ is a smooth manifold of dimension $n+1$, with boundary strata*

$$M \quad \text{and} \quad \bigsqcup_x \mathcal{M}(x) \times \mathcal{P}(x).$$

We have an evaluation map

$$\mathrm{ev} : \mathcal{R}(M) \to \mathcal{L}M,$$

which determines an orientation of $\mathcal{R}(M)$ relative η as follows: for a map $u \in \mathcal{R}(M)$, we have an isomorphism

$$|\det(D_u)| \cong \mathrm{ev}^*(\eta)$$

like in Equation (5.2.4).

Given any critical point y of f^r of index $n+1$, for generic data, the transverse fibre product

$$\mathcal{R}(M) \times_{\mathrm{ev}_r} W^s(y)$$

has dimension

$$\begin{aligned}\dim(\mathcal{R}(M)) + \dim(W^s(y)) - \dim(\mathcal{L}^r M) &= n+1+\dim(\mathcal{L}^r M) - (n+1) - \dim(\mathcal{L}^r M)\\ &= 0.\end{aligned}$$

Thus an element $(u,\gamma) \in \mathcal{R}(M) \times_{\mathrm{ev}_r} W^s(y)$ induces a canonical map

$$\mathcal{H}^e_{(u,\gamma)} : \mathbb{Z} \to o_y \otimes \eta_y$$

at the level of orientation lines. We use this to define a map

$$\mathcal{H}^e : \mathbb{Z} \to CM_{-1}(f^r;\eta) \tag{5.3.5}$$

$$\mathcal{H}^e(1) := \sum_{\substack{\mathrm{ind}(y)=n-1\\(u,\gamma)\in\mathcal{R}(M)\times_{\mathrm{ev}_r}W^s(y)}} \mathcal{H}^e_{(u,\gamma)}(1). \tag{5.3.6}$$

Remark 5.3.4. It is important to note that $CM_{-1}(f^r;\eta)$ is not necessarily zero since the grading includes the Morse index of y together with the degree in which the local system η is supported.

The boundary decomposition of $\mathcal{R}(M)$ then induces a cobordism which implies that $\mathcal{H}^e$ defines a homotopy in the square

$$\begin{array}{ccc}\mathbb{Z} & \longrightarrow & CF^*(H;\mathbb{Z})\\ \downarrow & & \downarrow\\ CM_*(f^1;\eta) & \xrightarrow[\iota^r]{} & CM_*(f^r;\eta).\end{array} \tag{5.3.7}$$

In a very similar vein, one can also show compatibility of $\mathcal{V}_r$ with the product, for the details, we refer to [Abo14, Sec. 4.4.3.]. We have

Lemma 5.3.5 ([Abo14, Lem. 4.4.10.]). *Given Hamiltonians H^1 and H^2, the following diagram commutes up to homotopy whenever r_1 and r_2 are sufficiently large:*

$$\begin{array}{ccc}CF^i(H^1;\mathbb{Z}) \times CF^j(H^2;\mathbb{Z}) & \longrightarrow & CF^{i+j}(H^0;\mathbb{Z})\\ \downarrow & & \downarrow\\ CM_{-i}(f^{r_1};\eta) \otimes CM_{-j}(f^{r_2};\eta) & \longrightarrow & CM_{-i-j}(f^{r_1+r_2};\eta).\end{array} \tag{5.3.8}$$

Combining Diagrams (5.3.7) and (5.3.8), we conclude:

Proposition 5.3.6. *The map $\mathcal{V}$ preserves units*

$$\mathcal{V} \circ e = e;$$

and products

$$\mathcal{V} \circ * = * \, (\mathcal{V} \otimes \mathcal{V}).$$

6 Viterbo's Theorem

We will split the proof of Viterbo's theorem into two parts: surjectivity and injectivity. We will start with the construction of a map

$$\mathcal{F} : H_*(\mathcal{L}M;\eta) \to SH^{-*}(T^*M;\mathbb{Z}) \tag{6.0.1}$$

such that the following holds:

THEOREM 6.0.1 ([Abo14, Thm. 5.1.1]). *The composition*

$$H_*(\mathcal{L}M;\eta) \xrightarrow{\mathcal{F}} SH^{-*}(T^*M;\mathbb{Z}) \xrightarrow{\mathcal{V}} H_*(\mathcal{L}M;\eta) \tag{6.0.2}$$

is an isomorphism.

We thus in particular have that:

COROLLARY 6.0.2 ([Abo14, Cor. 5.1.2]). *$\mathcal{V}$ is surjective and $\mathcal{F}$ is injective.*

In the background of this proof will be a different flavour of Floer theory lurking, namely Lagrangian Floer theory; this is due to the fact that Lagrangians naturally crop up in the analysis of asymptotes on boundary punctures of Riemannian surfaces with boundary. We will thus give a brief introduction of the most important concepts, but we refer to the standard references for Lagrangian Floer cohomology by Fukaya, Oh, Ohta and Ono [Fuk+09a; Fuk+09b] and [Sei08].

6.1 Chords, Maslov Index, and the Action

6.1.a) Hamiltonian Chords and Action

Consider a pair of Lagrangians L_0 and L_1 in T^*M, and let H be a time-dependent Hamiltonian, parametrised by $t \in [0,1]$.

DEFINITION 6.1.1. *A time-1* ***Hamiltonian chord*** *of X_H, starting on L_0 and ending on L_1 is a time-1 flow line of X_H*

$$x : [0,1] \to T^*M \tag{6.1.1}$$

$$\frac{dx}{dt} = X_{H_t} \tag{6.1.2}$$

whose endpoints lie on L_i:

$$x(i) \in L_i \quad \text{for} \quad i \in \{0,1\}.$$

In a way, Hamiltonian chords can be seen as the analogue of orbits for Lagrangian Floer theory. We will denote the collection of time-1 Hamiltonian chords by $\mathfrak{X}_H(L_0,L_1)$.

Denoting by Φ the time-1 Hamiltonian flow of H, there is an obvious bijection

$$\Phi(L_0) \cap L_1 \cong \mathfrak{X}_H(L_0,L_1).$$

Next, we define a notion of non-degeneracy.

Definition 6.1.2. *A chord $x \in \mathfrak{X}_H(L_0, L_1)$ is* ***non-degenerate*** *if the corresponding intersection point betwen $\Phi(L_0)$ and L_1 is transverse.*

We will only consider a very restricted set of Langrangians, specifically cotangent fibres and the zero section. This is partially due to the fact that these consist of a special type of Langrangians, exact Lagrangians, which are characterised by the fact that the primitive λ is exact on them.

Lemma 6.1.3. *The restriction $\lambda|_M$ and $\lambda|_{T^*M}$ both vanish identically.*

Proof. Let $v \in T_{(q,0)}M \subset T_{(q,0)}T^*M$. Then we have that

$$\begin{aligned}\lambda(v) &= 0|_q\big(D\pi(q,0)[v]\big)\\ &= 0.\end{aligned}$$

For $w \in T_{(q,p)}T^*_qM \subset T_{(q,p)}T^*M$, we have that $D\pi(q,p)[w] = 0 \in T_qM$, thus

$$\lambda(w) = 0.$$

□

Definition 6.1.4. *If $\lambda|_{L_i} \equiv 0$, we define the* ***action*** *of a chord $x \in \mathfrak{X}_H(L_0, L_1)$ as*

$$\mathcal{A}(x) := \int H \circ x dt \ - x^*(\lambda).$$

The critical points of $\mathcal{A}$ are constant paths at the intersection points $\Phi(L_0) \cap L_1$, and thus elements of $\mathfrak{X}_H(L_0, L_1)$. This action functional can be used to define Lagrangian (co)homology.

6.1.b) Strips, Half-Planes and Triangles

In order to define the map $\mathcal{F}$ described in the introduction, we will need to construct moduli spaces of punctured discs with cotangent boundary conditions, and then conclude with a cobordism argument. First, we will need to lay some groundwork and develop a way to grade and orient such spaces, just like in Section 2.4.

We start by considering the analogue of the cylinder for Hamiltonian chords, which is the cylinder:

$$B := \mathbb{R} \times [0,1] \tag{6.1.3}$$

equipped with the standard coordinates (s,t) and complex structure $j\partial_s = \partial_t$. We define the positive and negative half-strips to be:

$$B^+ := [0, +\infty) \times [0,1] \ \text{ and } \ B^- := (-\infty, 0] \times [0,1]. \tag{6.1.4}$$

Mimicking the approach of defining orientation lines of orbits, we define for each chord an operator on the half plane:

$$\mathbb{C}_+ := \{x + iy \mid y \geq 0\} \subset \mathbb{C}. \tag{6.1.5}$$

By default, we will equip $\mathbb{C}_+$ with the negative strip-like end

$$\varepsilon^- : B^- \to \mathbb{C}_+ \tag{6.1.6}$$
$$(s,t) \mapsto e^{-s-i\pi(1+t)}; \tag{6.1.7}$$

however, we will sometimes need to equip it with a positive end

$$\varepsilon^+ : B^+ \to \mathbb{C}_+ \tag{6.1.8}$$
$$(s,t) \mapsto e^{s+i\pi t};. \tag{6.1.9}$$

One of the models we will consider is the punctured Riemannian surface

$$T := B \setminus \{(0,0)\} \tag{6.1.10}$$

equipped with the canonical positive and negative strip-like ends $\varepsilon_{\pm\infty}$ which are given by the inclusions $B^\pm \subset B$. In addition, we endow the puncture at $(0,0)$ with a positive strip-like end $\varepsilon_{(0,0)}$ whose image is contained in $[-1,1] \times [0,1]$.

6.1.c) The Maslov Index for Paths

The goal of this section is to assign an index to a path in the Gassmannian of $\mathbb{C}^n$. This will help us resolve the issue that associating a $\mathbb{Z}$-graded determinant line to Hamiltonian chords requires a choice of a path in the Grassmannian of Lagrangians between the tangent spaces of the two Lagrangians corresponding to the two endpoints.

Fix a projection $t : \mathbb{C}_+ \to [0,1]$ which agrees with the t coordinate on the image of the strip-like end. Consider a pair of paths

$$\Lambda : [0,1] \to \mathrm{Gr}(\mathbb{C}^n)$$
$$A : [0,1] \to \mathrm{Sp}(2n, \mathbb{R})$$

such that $A_1\Lambda_0$ is transverse to $A_0\Lambda_1$.

Remark 6.1.5. Note that if A_t is constant, this reduces to the condition that Λ_0 is transverse to Λ_1.

Similar to Equation (2.4.3), fix any map

$$B^- : \mathbb{C}_+ \to \mathfrak{sp}(2n)$$

such that

$$B^-(\varepsilon^-(s,t)) = \frac{dA_t}{dt},$$

whenever $|s| \gg 0$. We obtain a Fredholm operator on the space of $\mathbb{C}^n$-valued functions with Λ as Lagrangian boundary condition:

$$\begin{aligned} D^-_{A,\Lambda} : W^{1,p}((\mathbb{C}_+, \mathbb{R}), (\mathbb{C}^n, \Lambda)) &\to L^p(\mathbb{C}_+, \mathbb{C}^n) \\ X &\mapsto \partial_x X + (I - B^-)\partial_y X. \end{aligned} \tag{6.1.11}$$

When A is constant, we write D_Λ for this operator.

Definition 6.1.6. *The **Maslov index** of the path Λ is the Fredholm index of the operator D_Λ.*

6.1.d) Orientation Lines of Hamiltonian Chords

We will be mostly interested in the Hamiltonian flow of functions of the form $h \circ \rho$, where ρ is the distance to the zero section. The most important example will be the Hamiltonian flow of $\frac{\rho^2}{2}$, which is:

$$\rho X_\rho.$$

Recall the map

$$\tilde{g} : T^*M \to TM.$$

The Hamiltonian $H := \frac{\rho^2}{2}$ can be used to identify the Hamiltonian flow of $\frac{1}{2}\rho^2$ and the geodesic flow in the following way: suppose that $(q^1, \dots, q^n, p_1, \dots, p_n)$ are the canonical coordinates on M. The Hamiltonian H can then be expressed as $H(q,p) = \frac{1}{2}g^{ij}(q)p^i p^j$. A curve $\gamma(t) := (q(t), p(t))$ is an integral curve of X_H if and only if it satisfies Hamilton's equations

$$\frac{d}{dt}q^i = g^{ij}p_j \quad \text{and} \quad \frac{d}{dt}p_i = -\frac{1}{2}(\partial_i g^{jk}(q))p_i p_k.$$

Now as $g_{ki}g^{il} = \delta^l_k$, we have $g_{ki}\partial x_m g^{il} = -g^{il}\partial x_m g_{ki}$ and multiplying by g^{jk} gives $\partial_m g^{jl} = -g^{jk}g^{il}\partial x_m g_{ki}$. Using that $p_i = g_{im}\dot{q}^m$, we get

$$\dot{p}_i = \frac{1}{2}\partial_i g_{sr}\dot{q}^s\dot{q}^r.$$

We thus have

$$\begin{aligned}
\ddot{q}^l &= g^{im}p_m + \partial_j g^{ik}\dot{q}^j p_k \\
&= \frac{1}{2}g^{im}[\partial_m g_{jk} - 2\partial_j g_{mk}]\dot{q}^j\dot{q}^k \\
&= -\Gamma^i_{jk}\dot{q}^j\dot{q}^k,
\end{aligned}$$

which is exactly the geodesic equation. A similar computation then also shows the converse, i.e. that solutions to the geodesic equation also satisfy Hamilton's equation.

The following result follows immediately from the above observation:

LEMMA 6.1.7 ([Abo14, Lem. 5.2.7.]). *Projection to M defines a bijection between (i) time-1 chords of bX_ρ, starting on $T^*_{q_0}M$ and ending on $T^*_{q_1}M$, which are contained in a non-zero level set of ρ and (ii) based geodesics of length b starting at q_0 and ending at q_1.*

Since $\frac{\rho^2}{2}$ is not linear at infinity, we will consider a Hamiltonian which agrees with it near the zero section, and with an affine function of ρ away from D^*_2M. Formally, choose h such that

- $h(\rho) = \begin{cases} \frac{\rho^2}{2} & \text{if } \rho \in [0,1] \\ 2\rho - 2 & \text{if } 2 \leq \rho. \end{cases}$
- $\frac{d^2h}{d\rho^2}|_{(0,2)} > 0.$

Given a number $b \in [1,2]$, we can link elements of $\mathfrak{X}_{bh}(T^*_{q_0}M, T^*_{q_1}M)$ to geodesics from q_0 to q_1. To see this, note that the Hamiltonian flow of bh preserves the level sets of ρ, and its restriction to a given level set agrees with the flow of

$$b\frac{dh}{d\rho}X_\rho.$$

This flow agrees, up to rescaling, with the flow of $\frac{\rho^2}{2}$, which is ρX_ρ. It follows that the time-1 lines of bh which are contained in this level set project to geodesics of length $b\frac{dh}{d\rho}$. Since the derivative of h increases monotonically from 0 to 2 in the interval $[0,2]$, we can relate chords to geodesics.

LEMMA 6.1.8 ([Abo14, Lem. 5.2.10.]). *There is a bijection from $\mathfrak{X}_{bh}(T^*_{q_0}M, T^*_{q_1}M)$ contained in D^*_2M and the set of geodesics between q_0 and q_1 of length bounded by $2b$.*

Recalling our standing assumption (H1) that the injectivity radius of our chosen metric on M is larger than 4, we have:

COROLLARY 6.1.9 ([Abo14, p. 5.2.11.]). *Given a constant $b < 2$, there is a unique chord in $\mathfrak{X}_{bh}(T^*_{q_0}M, T^*_{q_1}M)$ if $d(q_0, q_1) < 2b$.*

We can now combine this with the results from Section 6.1.d) to orient lines of Hamiltonian chords. Let $x : [0,1] \to T^*M$ be a Hamiltonian chord of bh for $b \in [1,2]$ with endpoints in $T^*_{q_0}M$ and $T^*_{q_1}M$. By pulling back TT^*M by $x \circ t$, we obtain a symplectic vector bundle over $\mathbb{C}_+$. The bundle is then equipped with the Lagrangian boundary conditions

$$T^*_{\pi(x(t))}M.$$

The image of these Lagrangians under a trivialisation of $(x \circ t)^*(TT^*M)$ then serves as a path $\Lambda^x : [0,1] \to \mathrm{Gr}(\mathbb{C}^n)$. As in Section 2.4.b), let $A^x_t \in \mathfrak{sp}(2n)$ be the path of matrices such that $\psi_t = \exp(A^x_t)$, where ψ_t is the flow induced by the Hamiltonian h. By Equation (6.1.11), we obtain an operator

$$D_{A^x,\Lambda^x} : W^{1,p}((\mathbb{C}_+, \mathbb{R}), (\mathbb{C}^n, \Lambda^x)) \to L^p(\mathbb{C}_+, \mathbb{C}^n).$$

We then define

DEFINITION 6.1.10. *The **orientation line** associated to a Hamiltonian chord x is defined as*

$$o_x := \det(D_{A^x,\Lambda^x}).$$

We have the following result when the endpoints of the Hamiltonian chord are close together.

LEMMA 6.1.11 ([Abo14, Lem. 5.2.26.]). *If x corresponds to a geodesic which is shorter than the injectivity radius, the index of D_{A^x,Λ^x} vanishes, and o_x admits a canonical trivialisation.* $\square_{\text{ref.}}$

6.1.e) Integrated Maximum Principle for Riemannian Surfaces with Boundary

In this section, we will discuss a generalisation of the procedure used to compactify the moduli space $\mathcal{M}(x_0, x_1)$ in Corollary 2.3.11. We will see that the image of pseudo-holomorphic maps from a Riemannian surface Σ to T^*M must lie in the interior of D_l^*M — given certain choices of Floer data and boundary conditions. In future arguments, this result, that we shall call the integrated maximum principle, will be used to show a plethora of compactness results.

Let $\overline{\Sigma}$ be a compact Riemannian surface with boundary such that the set $\overline{\Sigma}\backslash\Sigma = \{z_i\}_{i=1}^e$ is finite. Fix a decomposition of the boundary into manifolds with boundary

$$\partial\Sigma = \partial^n\Sigma \cup \partial^l\Sigma,$$

such that $\partial^n\Sigma \cap \partial^l\Sigma$ consists of finitely many points, and the ends of Σ are disjoint from $\partial^n\Sigma$. The superscripts stand for *normal* and *Lagrangian* boundary conditions.

This decomposition will naturally appear in applications in the following way. First, we define Σ to be the inverse image of $T^*M\backslash D_l^*M$ of some pseudo-holomorphic map that we want to analyse. The normal component $\partial^n\Sigma$ will then correspond to the inverse image of S_l^*M, while the other boundary maps to a collection of Lagrangians.

We now make the assumptions on the Floer data precise. At each puncture z_i, we fix negative strip-like ends

$$\varepsilon_i : B^- \to \Sigma.$$

On the space of maps

$$u : \Sigma \to T^*M \setminus D_l^*M,$$

consider the pseudo-holomorphic equation

$$(Du - \sum_{j=0}^{k} \alpha_j \otimes X_{h_j})^{(0,1)} = 0, \tag{6.1.12}$$

where α_j are 1-forms on Σ, h_j are functions that only depend on ρ^2, in addition to a family J_z of almost complex structures on T^*M. We pose the following conditions on the Floer data:

(6.1.13.i) $\varepsilon_i^*(\alpha_j) = dt$ for all $1 \leq i \leq e$ and $0 \leq j \leq k$,

(6.1.13.ii) $\alpha_j|_{\partial^l\Sigma} = 0$ for all $0 \leq j \leq k$,

(6.1.13.iii) J_z is convex near S_l^*M,

(6.1.13.iv) each h_j is non-decreasing and convex,

(6.1.13.v) $d\alpha_j \leq 0$ for all $0 \leq j \leq k$.

Moreover, we require that u satisfies the normal boundary condition:

(H3) u is transverse to S_l^*M and $u^{-1}(S_l^*M) = \partial^n\Sigma$;

together with the following condition on the Lagrangian boundary: for each component $I \subset \partial^l\Sigma$, choose a point $q_I \in M$. The condition along $\partial^l\Sigma$ then states as:

(H4) $u(I) \subset T^*_{q_I}M$.

We can now generalise the notion of energy to:

$$E_{\text{geo}}(u) := \frac{1}{2}\int_\Sigma ||Du - \sum_{j=0}^{k} \alpha_j \otimes X_{h_j}||^2. \tag{6.1.14}$$

LEMMA 6.1.12 ([Abo14, Lem. 5.2.29.]). *If $E_{\text{geo}}(u)$ vanishes, the image of u is contained in a (not necessarily closed) orbit of X_{ρ^2}.*

Proof. Since $||Du - \sum_{j=0}^{k} \alpha_j \otimes X_{h_j}||^2 \geq 0$, in order for E_{geo} to vanish, we need that $||Du - \sum_{j=0}^{k} \alpha_j \otimes X_{h_j}||^2$ vanishes at every point, implying that $Du - \sum_{j=0}^{k} \alpha_j \otimes X_{h_j}$ vanishes at every point. We hence have that the image of Du is contained in the subspace spanned by the vector fields X_{h_j}; explicitly, given $\sigma \in \Sigma$ together with a tangent vector $v \in T_\sigma\Sigma$, we have

$$Du(\sigma)[v] = \sum_{j=0}^{k} \alpha_j(v) \otimes X_{h_j}(u(\sigma)).$$

Each h_j is a function of ρ^2 and thus we see that $Du(\sigma)[v] = \lambda_v \cdot X_{\rho^2}(u(\sigma))$. We conclude that the image of u is contained in an orbit of X_{ρ^2}. □

We can use the energy to make statements of the strip-like ends:

LEMMA 6.1.13 ([Abo14, Lem. 5.2.30.]). *If $E_{\text{geo}}(u)$ is finite, the limit*

$$\lim_{s\to-\infty} u(\varepsilon_i(s,t))$$

is well-defined, and converges to a time-1 chord x_i of the Hamiltonian

$$h := \sum_{j=1}^{k} h_j$$

with end points on cotangent fibres.

Proof. Equation (6.1.13.i) implies that :

$$\begin{aligned} 0 \overset{!}{=} (D(u\circ\varepsilon_i) - \sum \alpha_j \otimes X_{h_j})^{0,1} &= (D(u\circ\varepsilon_i) - dt \otimes \sum X_{h_j})^{0,1} \\ &= (D(u\circ\varepsilon_i) - dt \otimes X_h)^{0,1}, \end{aligned}$$

where the last line used the additivity of Hamiltonian vector fields. Moreover, since $\partial^n\Sigma$ was assumed to be disjoint from the punctures, the two connected components of the boundary of the strip are mapped to two cotangent fibres. We have then reduced the problem to a standard result in Lagrangian Floer theory. As for example stated in [Flo88d, Thm. 2], finite energy strips with Lagrangian boundary condition converge to Hamiltonian chords. □

We are now in a position to state the integrated maximum principle.

PROPOSITION 6.1.14 ([Abo14, Prop. 5.2.35.]). *There is no finite energy map satisfying Equation (6.1.12) with boundary condition satisfying (H3) and (H4), if the Floer data in addition satisfies Equations (6.1.13.i)-(6.1.13.v).*

We will also briefly need a second notion of energy in an upcoming computation; this energy is defined without reference to a metric and it is called the topological energy:

$$E_{\text{top}} = \int_\Sigma u^*(\omega) - \sum_{j=0}^{k} \int_\Sigma d(u^*(h_j)\alpha_j). \tag{6.1.15}$$

It can be expressed in terms of the geometric energy:

LEMMA 6.1.15 ([Abo14, Lem. 5.2.31.]). *The geometric and topological energy are related by the following equation*

$$E_{\text{geo}}(u) = E_{\text{top}}(u) + \sum_{j=0}^{k} \int_\Sigma u^*(h_j) d\alpha_j. \tag{6.1.16}$$

By Lemma 6.1.13, the map u converges to chords x_i at the strip-like ends. As it turns out, the topological energy can then alternatively be expressed in terms of the action of these Hamiltonian chords.

LEMMA 6.1.16 ([Abo14, Lem. 5.2.33.]). *The topological energy of u is given by:*

$$E_{\text{top}}(u) = \int_{\partial^n \Sigma} u^*(\lambda) - \sum_{j=0}^{k} \int_{\partial^n \Sigma} u^*(h_j)\alpha_j + \sum_{i=1}^{e} \mathcal{A}(x_i). \tag{6.1.17}$$

$\square_{\text{ref.}}$

6.2 From Morse Homology to Floer Cohomology

At the heart of the construction of a map

$$\mathcal{F} : H_*(\mathcal{L}M; \eta) \to SH^*(T^*M; \mathbb{Z}) \tag{6.2.1}$$

lies a map

$$\mathcal{F}^r : H_*(\mathcal{L}^r M; \eta) \to HF^{-*}(H^-; \mathbb{Z}) \tag{6.2.2}$$

commuting with all the relevant maps such that the colimit constructions yield the desired map $\mathcal{F}$. The subsequent section will be dedicated to defining such a map.

6.2.a) Moduli Space of Punctured Discs with Cotangent Boundary Conditions

As always, we will begin by constructing a suitable moduli space by considering appropriate maps from punctured discs with Lagrangian boundary conditions similar to those of Section 5.2. The main novelty will be that the puncture in the interior

will carry a negative end and that we will consider multiple boundary punctures with positive ends.

Let Z^- be the punctured disc that we, as the notation indicates, identify with the negative half cylinder:

$$Z^- := (-\infty, 0] \times S^1. \tag{6.2.3}$$

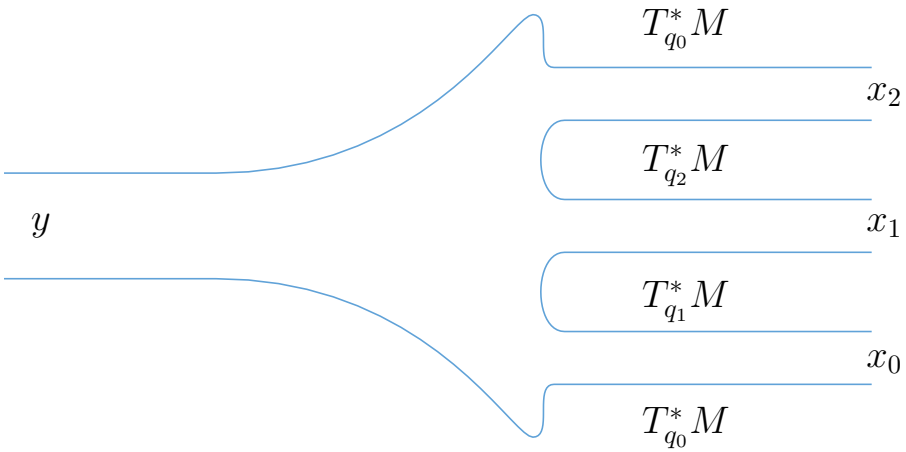
Given a point ξ on the boundary of $\mathbb{Z}^-$, we define a positive strip-like end at ξ to be a map:

$$\begin{aligned} \varepsilon_\xi : B^+ &\to Z^- \\ \lim_{s\to+\infty} \varepsilon_\xi(s,t) &= \xi \end{aligned}$$

which we assume to be biholomorphic onto a punctured neighbourhood of ξ. Fix a positive integer r[9], and denote by Z_r^- the complement of r uniformly distributed points on the boundary:

$$Z_r := Z^- \setminus \{(0, \frac{i}{r}\}_{i=1}^r. \tag{6.2.4}$$

We then fix positive strip-like ends $\{\varepsilon_i\}_{i=1}^r$ at the points $(0, \frac{i}{r})$ whose images are disjoint.



Next, we define the pseudo-holomorphic equation whose solutions constitute the moduli space. Let H_z and J_z^- be families of Hamiltonians and complex structures on T^*M that are parametrised by points $z \in \mathbb{Z}_r^-$. Additionally, let α^- be a 1-form on $\mathbb{Z}_r^-$, and consider the pseudo-holomorphic equation:

$$(Du - \alpha^- \otimes X_H)^{0,1} = 0 \tag{6.2.5}$$

for $u : \mathbb{Z}_r^- \to T^*M$. To achieve Lagrangian boundary conditions, choose a point $\vec{q} \in \mathcal{L}^r M$ and require that

$$u\Big(0 \times \Big[\frac{i-1}{r}, \frac{i}{r}\Big]\Big) \subset T^*_{q_i} M. \tag{6.2.6}$$

As usual, we impose some restrictions on the Floer data. Recall that in definition 4.2.3, we made a choice of constants δ_i^r that bound the length of the i-th segment

[9]This integer will later on correspond to the number of factors of M in $\mathcal{L}^r M$.

of a piecewise geodesic from q_i to q_{i+1}. Fix constants b_i such that

$$\delta_i^r < 2b_i < 2.$$

Applying Corollary 6.1.9 to each individual geodesic connecting the points of $\vec{q} \in \mathcal{L}^r M$, we obtain a unique Hamiltonian chord x_i of $b_i h$ with endpoints in $T^*_{q_i} M$ and $T^*_{q_{i+1}} M$.

Define $b := \sum_i b_i$. First, we restrict the 1-form α^- on $\mathbb{Z}_r^-$:

(H5) α^- is closed. The restriction to a subset of $\mathbb{Z}_r^-$ with $s \leq -1$ agrees with b dt, the restriction of $\partial \mathbb{Z}_r^-$ vanishes identically, and the pullback $\varepsilon_i^*(\alpha^-) \equiv b_i dt$.

Furthermore, we choose a constant b^- which is sufficiently larger such that:

$$2b < b^-.$$

We are now in a position to state the restrictions on the family of Hamiltonians:

(6.2.7.i) the restriction of H to a neighbourhood of the boundary agrees with the model Hamiltonian:

$$H_{(s,t)} \equiv h \quad \text{if } s \geq -1.$$

(6.2.7.ii) there exists a Hamiltonian H_t^+ of slope b^-, all of whose orbits are non-degenerate, such that

$$H_{(s,t)} = \frac{H_t}{b} \quad \text{if } 0 \gg s.$$

(6.2.7.iii) there exists a function f^- on Σ such that:

$$H_z|_{T^*M \setminus D_2^* M} \equiv 2\rho - 2 + f^-\Big(\frac{b^-}{b}\rho - 2\rho + 2\Big).$$

This function f^- shall in addition satisfy:

$$\begin{aligned} \frac{\partial f^-}{\partial s} &\leq 0 \\ f^-(s,t) &= 0 \quad \text{if } -1 \leq s \\ f^-(s,t) &= 1 \quad \text{if } s \ll 0. \end{aligned}$$

Lemma 6.2.1. *Let u be a finite energy solution of (6.2.5). Then there exists an orbit $x \in \mathcal{O}(H^-)$ together with Hamiltonian chords $x_i \in \mathfrak{X}_{b_i h}(T^*_{q_i} M, T^*_{q_{i+1}} M)$ such that:*

$$\begin{aligned} \lim_{s \to -\infty} u(s,t) &= x(t), \\ \lim_{s \to +\infty} u(\varepsilon_i(s,t)) &= x_i(t). \end{aligned}$$

Proof. We are going to look at the case $s \to +\infty$. The other one follows from a slight modification of Lemma 2.3.8. In the case $s \geq -1$, Equation (6.2.5) reduces to $(D(u \circ \varepsilon_i) - dt \otimes b_i h)^{0,1}$ on the strip-like end ε_i. The boundary of the strip is mapped to two cotangent fibres under Assumption (6.2.6) and thus we can again employ [Flo88d, Thm. 2] to conclude. □

We will write

$$\mathcal{M}(x, \vec{q})$$

for the set of solution to (6.2.5).

Remark 6.2.2. This space is arguably the most important moduli space of the entire proof of Viterbo's theorem since it ultimately links orbits to piecewise geodesics. It thus provides a powerful link between Floer (co-)homology and loop homology.

If in addition, we assume that the family of complex structures is convex near S_2^*M, we obtain the following compactness result, which is our first application of the integrated maximum principle.

Lemma 6.2.3 ([Abo14, Lem. 5.3.5.]). *Under the above assumptions, all elements of $\mathcal{M}(x, \vec{q})$ have image contained in D_2^*M.*

Proof. All of these types of proofs will follow the mould of the archetypal proof of Lemma 2.3.7. Let $u \in \mathcal{M}(x, \vec{q})$ and assume by contradiction that its image is not contained in D_2^*M. In particular, the inverse image of D_l^*M is non-empty for $l \geq 2$ sufficiently close to 2. The solution u intersects S_l^*M for a generic choice of l, and the inverse image $\Sigma := u^{-1}(T^*M \setminus D_l^*M)$ is non-empty. We need the intersection to be transversal so that Σ becomes a compact Riemannian surface with boundary $u^{-1}(S_l^*M)$.

Since we are now working outside of D_2^*M, the Hamiltonian takes on the form

$$H_z \equiv 2\rho - 2 + f^-\Big(\frac{b^-}{b}\rho - 2\rho + 2\Big)$$

by Equation (6.2.7.iii). The pseudo-holomorphic equation of (6.2.5) then becomes

$$\Big(Du - \Big(2 + \Big(\frac{b^-}{b} - 2\Big)f^-\Big)\alpha^- \otimes X_\rho\Big)^{0,1} = 0,$$

where the constant terms of the Hamiltonian disappeared since the associated Hamiltonian flow vanishes. All that is left is to show that $d\Big(\Big(2+\Big(\frac{b^-}{b}-2\Big)f^-\Big)\alpha^-\Big) \leq 0$ in order to apply Proposition 6.1.14. Since $\frac{b^-}{b} - 2$ is positive by assumption, it is enough to check that $d(f^-\alpha^-) \leq 0$. For $s \geq -1$, the form α^- is closed and f^- is constant, which implies that we only need to concern ourselves with the case where $s \leq -1$. In this region, α^- is a constant multiple of dt:

$$\begin{aligned} d(f^-\alpha^-) &= d(bf^-dt) \\ &= b\ df^- \wedge dt \\ &= b\ \Big(\frac{\partial f^-}{\partial s}ds \wedge \frac{\partial f^-}{\partial t}dt\Big) \wedge dt \\ &= b\frac{\partial f^-}{\partial s}ds \wedge dt. \end{aligned}$$

Since $\frac{\partial f^-}{\partial s} \leq 0$ by ((6.2.7.iii)), we have that the 2-form in non-positive. Proposition 6.1.14 then yields that Σ is empty. Since the constant l was generically chosen, we can take l to be arbitrarily close to 2, implying that the image of u is contained in $D_2^* M$. □

6.2.b) The Space $\mathcal{M}(x, \mathcal{L}^r M)$

Roughly speaking, the map

$$\mathcal{F}^r : H_*(\mathcal{L}^r M; \eta) \to HF^{-*}(H^-; \eta)$$

will be constructed by taking a, by orientations, weighted sum over all the possible spaces $\mathcal{M}(x, \vec{q})$. We thus need to consistently orient the spaces $\mathcal{M}(x, \vec{q})$. A gluing construction yields the result:

LEMMA 6.2.4 ([Abo14, Lem. 5.3.9.]). *The virtual dimension of $\mathcal{M}(x, \vec{q})$ is $\deg(x) - n$. Moreover, for every element of this moduli space, there is a canonical isomorphism*

$$|\det(D_u)| \cong o_x \otimes \eta_{\vec{q}}[w(x)], \tag{6.2.8}$$

where D_u is the linearisation of Equation (6.2.5).

□ref.

As a map $\mathcal{F}^r$, needs to take into account all possibly piecewise geodesics end, define

$$\mathcal{M}(x, \mathcal{L}^r M) := \bigsqcup_{\vec{q} \in \mathcal{L}^r M} \mathcal{M}(x, \vec{q}).$$

This space admits a natural topology as a parametrised moduli space by embedding into

$$C^\infty(Z_r^-, T^*M) \times \mathcal{L}^r M.$$

The following projection onto the parameter space will come in handy later on

$$\pi_{\vec{q}} : \mathcal{M}(x, \mathcal{L}^r M) \to \mathcal{L}^r M.$$

For generic Floer data, a solution $u \in \mathcal{M}(x, \mathcal{L}^r M)$ can be assumed to be regular, implying that $\mathcal{M}(x, \mathcal{L}^r M)$ is a smooth manifold.

LEMMA 6.2.5. *The dimension of $\mathcal{M}(x, \mathcal{L}^r M)$ is $(r-1)n + \deg(x)$. Furthermore, there is a canonical isomorphism*

$$|\mathcal{M}(x, \mathcal{L}^r M)| \cong o_x[w(x)] \otimes \eta_{\vec{q}} \otimes |T_{\vec{q}} \mathcal{L}^r M| \tag{6.2.9}$$

at every point $u \in \mathcal{M}(x, \mathcal{L}^r M)$ with $\vec{q} = \pi_{\vec{q}}(u)$.

Proof. Since $\mathcal{M}(x, \mathcal{L}^r M)$ inherits its topology by its interpolation as a parametrised moduli space, we have that

$$T_u \mathcal{M}(x, \mathcal{L}^r M) \cong T_u \mathcal{M}(x, \vec{q}) \oplus T_{\pi_{\vec{q}}(u)} \mathcal{L}^r M.$$

At the level of orientation lines, this induces the isomorphism

$$\begin{aligned}|\mathcal{M}(x,\mathcal{L}^r M)| &\cong |\mathcal{M}(x,\vec{q})| \otimes |T_{\vec{q}}\mathcal{L}^r M| \\ &\overset{(6.2.8)}{\cong} o_x \otimes \eta_{\vec{q}}[w(x)] \otimes |T_{\vec{q}}\mathcal{L}^r M|.\end{aligned}$$

The dimension of $\mathcal{M}(x,\mathcal{L}^r M)$ is then simply a matter of adding up the degrees:

$$\begin{aligned}\dim(\mathcal{M}(x,\mathcal{L}^r M)) &= |x| - 2n + n - w(x) + rn \\ &= \deg(x) - n + rn \\ &= \deg(x) + (r-1)n.\end{aligned}$$

□

6.2.c) Construction of $\mathcal{F}^r$

Since of course not every point of $\mathcal{L}^r M$ is a critical point of $\mathcal{L}^r M$, we first construct a new space that connects $\mathcal{M}(x,\vec{q})$ to critical points. Let y be a critical point of a Morse function f^r on $\mathcal{L}^r M$, and let x be a Hamiltonian orbit of a non-degenerate linear Hamiltonian H^- whose slope is larger than $2r$.

The space that we use to make this transition is the following fibre product:

$$\mathcal{B}(x,y) := \mathcal{M}(x,\mathcal{L}^r M) \times_{(\pi_{\vec{q}},\iota;\mathcal{L}^r M)} W^u(y), \tag{6.2.10}$$

where we specified the maps that go into the fibre product.

Lemma 6.2.6. *For generic data, the space $\mathcal{B}(x,y)$ is a smooth manifold of dimension*

$$\operatorname{ind}(y) + \deg(x) - n$$

and there exists a canonical isomorphism

$$|\mathcal{B}(x,y)| \cong o_x[w(x)] \otimes \eta_y \otimes o_y. \tag{6.2.11}$$

Proof. For generically chosen data, the intersection $\pi_{\vec{q}}(\mathcal{M}(x,\mathcal{L}^r M)) \cap W^u(y)$ is transverse, and thus $\mathcal{B}(x,y)$ is a smooth manifold of dimension

$$\begin{aligned}\dim(\mathcal{M}(x,\mathcal{L}^r M) + \dim(W^u(y)) - \dim(\mathcal{L}^r M) &= \deg(x) + (r-1)n + \operatorname{ind}(y) - nr \\ &= \deg(x) + \operatorname{ind}(y) - n.\end{aligned}$$

As for the statement about orientation lines, we start with the short exact sequence

$$T\mathcal{B}(x,y) \longrightarrow T\mathcal{M}(x,\mathcal{L}^r M) \oplus TW^u(y) \longrightarrow T\mathcal{L}^r M$$

which yields the isomorphism

$$|\mathcal{B}(x,y)| \otimes |T_{\vec{q}}\mathcal{L}^r M| \cong |\mathcal{M}(x,\mathcal{L}^r M)| \otimes |W^u(y)|.$$

Substituting in Equation (6.2.9), we obtain

$$T_{\vec{q}}\mathcal{L}^r M| \otimes |\mathcal{B}(x,y)| \cong o_x[w(x)] \otimes \eta_y \otimes |T_{\vec{q}}\mathcal{L}^r M| \otimes o_y.$$

□

When $\dim(\mathcal{B}(x,y)) = 0$, i.e. if $\deg(x) + \mathrm{ind}(y) - n = 0$, Equation (6.2.11) yields the isomorphism

$$\mathcal{F}^u : o_y \otimes \eta_y \cong o_x[w(x)]$$

since in this case we have $\mathbb{Z} \cong o_x[w(x)] \otimes \eta_y \otimes o_y$. Lemma 6.2.3, together with Gromov-Floer compactness imply that $\mathcal{B}(x,y)$ is compact, and hence finite as a 0-dimensional manifold. Define

$$\mathcal{F}^r : CM_*(f^r;\eta) \to CF^{-*}(H^-;\mathbb{Z}) \tag{6.2.12}$$

$$\mathcal{F}^r\,|_{o_y\otimes\eta_y} := \bigoplus_{\substack{\deg(x)+\mathrm{ind}(y)-n=0 \\ u\in\mathcal{B}(x,y)}} (-1)^{\mathrm{ind}(y)-n}\,\mathcal{F}^u\,. \tag{6.2.13}$$

The additional sign ensures that $\mathcal{F}^r$ is a chain map. The following proof is inspired by the proof of Lemma 5.2.6.

Lemma 6.2.7. *The map $\mathcal{F}^r$ is a chain map, i.e.*

$$d \circ \mathcal{F}^r = \mathcal{F}^r \circ \partial.$$

Proof (Sketch). Consider the moduli space $\mathcal{B}(x,y)$ which is 1-dimensional. The natural compactification of this space is a manifold $\overline{\mathcal{B}}(x,y)$ with boundary

$$\bigsqcup_{\deg(x)=\deg(x_0)-1} \mathcal{M}(x,x_0) \times \mathcal{B}(x_0,y) \tag{6.2.14}$$

$$\bigsqcup_{\mathrm{ind}(y)=\mathrm{ind}(y_1)+1} \mathcal{B}(x,y_1) \times \mathcal{T}(y_1,y). \tag{6.2.15}$$

In particular, we have that $\dim(\mathcal{B}(x_0,y)) = \dim(x,y_1) = 0$, thus there are isomorphisms

$$o_y \otimes \eta_y \cong o_{x_0}[w(x_0)] \quad \text{, and} \tag{6.2.16}$$

$$o_{y_1} \otimes \eta_{y_1} \cong o_x[w(x)]. \tag{6.2.17}$$

Given $u \in \mathcal{M}(x,x_0)$, the map d_u is induced by the natural isomorphism

$$\begin{aligned} o_x[w(x)] &\cong |\partial_s u| \otimes o_{x_0}[w(x_0)] \\ &\overset{(6.2.16)}{\cong} |\partial_s u| \otimes o_y \otimes \eta_y. \end{aligned} \tag{6.2.18}$$

In order to consider the other component of the boundary, note that an element $\gamma \in \mathcal{T}(y_1,y)$ is induced by the isomorphism

$$\begin{aligned} |\mathbb{R}\cdot\partial_s\gamma|^{-1} \otimes o_y &\cong o_{y_1} \\ &\overset{(6.2.17)}{\cong} o_x[w(x)] \otimes \eta_y^{-1}. \end{aligned} \tag{6.2.19}$$

Comparing (6.2.18) and (6.2.19) with the natural isomorphism

$$|\mathcal{B}(x,y)| \cong o_x[w(x)] \otimes \eta_y \otimes o_y$$

of Equation (6.2.11), and keeping track of Koszul signs, we can identify (6.2.14) with $d \circ \mathcal{F}^r$ and (6.2.15) with $-\mathcal{F}^r \circ \partial$. The proof then concludes by noticing that the orientations at the two components of $\partial\overline{\mathcal{B}}(x,y)$ must vanish since $\mathcal{B}(x,y)$ is a 1-dimensional manifold; thus we have:

$$d \circ \mathcal{F}^r - \mathcal{F}^r \circ \partial = 0.$$

□

LEMMA 6.2.8. *At the level of homology, the map $\mathcal{F}^r$ is independent of the choice of Floer data. Furthermore, there is a commutative diagram*

$$\begin{array}{ccc} HM_*(f^r;\eta) & \xrightarrow{\mathcal{F}^r} & HF^{-*}(T^*M;H^+) \\ & \searrow^{\mathcal{F}^r} & \downarrow \mathfrak{c} \\ & & HF^{-*}(T^*M;H^-). \end{array}$$

Proof (Sketch). We are only going to sketch why the diagram commutes. The statement about independence from the Floer data are proved analogously to Lemma 2.6.9.

One can build a family of Cauchy-Riemann equations on the negative punctured disc Z^- interpolating between (6.2.5) and the result of gluing this equation to the continuation map from H^- to H^+ on the cylinder.

We choose all almost complex structures to be convex near the unit sphere bundle, and the Hamiltonian so that for any fixed equation, the slope does not increase along the s coordinate of the negative punctured disc.

Let $(H_{s,t})_{s\in\mathbb{R},t\in S^1}$ be a family of interpolating Hamiltonians between H^- and H^+, and fix $\vec{q} \in \mathcal{L}^r M$. Denote by $\mathcal{K}(x,\vec{q})$ the space of solutions to the family of equations described above. The solutions of $\mathcal{K}(x,\vec{q})$ have cotangent boundary conditions at $(0,t)$ inherited from Equation (6.2.6) and asymptotic condition a Hamiltonian orbit x.

By construction, one boundary component of the Gromov-compactification $\overline{\mathcal{K}}(x,\vec{q})$ is the space $\mathcal{M}(x,\vec{q})$. The other end of the moduli space breaks into two components, due to the nature of the Gromov compactification of the continuation space, we have:

LEMMA 6.2.9. *For a regular choice of data, the moduli space $\mathcal{K}(x,\vec{q})$ is a smooth manifold. The codimension 1 strata of its compactification are:*

$$\begin{aligned} &\mathcal{M}(x,\vec{q}), \\ &\bigsqcup_{x_-\in\mathcal{O}(H^-)} \mathcal{K}(x,x_-)\times\mathcal{M}(x_-,\vec{q}), \\ &\bigsqcup_{x_+\in\mathcal{O}(H^+)} \mathcal{M}(x,x_+)\times\mathcal{K}(x_+,\vec{q})). \end{aligned}$$

□

As before, we then define

$$\mathcal{K}(x, \mathcal{L}^r M) := \bigsqcup_{\vec{q}} \mathcal{K}(x, \vec{q}).$$

An orientation of $\mathcal{K}(x, \mathcal{L}^r M)$ is then defined by an orientation of the interval used to parametrise the homotopy between the equation at the start, the fact that $\mathcal{L}^r M$ possesses an orientation induced by M, and lastly a choice of generators of o_x and η.

For r large enough and generically chosen data, the fibre product

$$\mathcal{K}(x, \mathcal{L}^r M) \times_{\mathcal{L}^r M} W^u(y)$$

defines a 1-dimensional cobordism between

$$\mathcal{B}(x, y), \tag{6.2.20}$$

$$\bigsqcup_{\substack{x' \\ \deg(x)=\deg(x')}} \mathcal{K}(x, x') \times \mathcal{B}(x', y) \tag{6.2.21}$$

whenever $\deg(x') + \operatorname{ind}(y) = n$. Elements of (6.2.20) correspond to $\mathcal{F}^r$, meanwhile elements of (6.2.21) correspond to $\mathfrak{c} \circ \mathcal{F}^r$. □

In addition to being compatible with continuation maps, a cobordism argument presented in [Abo14, Sec. 5.4.] yields:

Lemma 6.2.10. *Let $r > 0$. Whenever the slope of H is larger than $2r$, the following diagram commutes:*

$$\begin{array}{ccc} HM_*(f^{r-1}; \eta) & \xrightarrow{\iota} & HM_*(f^r; \eta) \\ & \searrow^{\mathcal{F}^{r-1}} & \downarrow{\scriptstyle \mathcal{F}^r} \\ & & HF^{-*}(H; \mathbb{Z}). \end{array}$$

An immediate consequence of Lemma 6.2.6, 6.2.8 and 6.2.10 is the following:

Corollary 6.2.11. *The maps $\mathcal{F}^r$ of Equation (6.2.12) induce a map*

$$\mathcal{F} : H_*(\mathcal{L}M; \eta) \to HF^{-*}(T^*M; \mathbb{Z}) \tag{6.2.22}$$

that is independent of all choices.

6.3 Composition on Loop Homology

We have now arrived at the heart of the surjectivity part of the proof of Viterbo's theorem. The goal of this section is to show that the composition $\mathcal{V}_{r'} \circ \mathcal{F}^r$, with certain restrictions on r' and r, is equal to the composition of $r' - r$-copies of ι up to a sign. As it will turn out, this sign depends on the orientability of the loops that are considered. Given the Morse homology of $\mathcal{L}^r M$, the following splitting appears naturally in the proof:

$$HM_*(f^r; \eta) \cong HM_*^0(f^r; \eta) \oplus HM_*^{-1}(f^r; \eta),$$

where the first direct summand corresponds to the components of piecewise geodesics along which TM is orientable, and the second summand to the components which are not. Note that both $\mathcal{V}_{r'}$ and $\mathcal{F}^r$ are defined using moduli spaces of punctured maps. These maps therefore preserves the decomposition of Morse homology and Floer cohomology, thus we have:

$$HM_*^w(f^r;\eta) \xrightarrow{\mathcal{F}^r} HF_w^{-*}(H^-;\mathbb{Z}) \xrightarrow{\mathcal{V}_{r'}} HM_*(f^{r'};\eta)$$

with $w \in \{0,1\}$ depending on the orientability, and when the slope of H^- is larger than $2r$ and r' sufficiently big.

The reasonings of this section will culminate in the proof of:

Proposition 6.3.1 ([Abo14, Prop. 5.5.1.]). *If r' is sufficiently large, the composition*

$$\mathcal{V}_{r'} \circ \mathcal{F}^r : HM_*^w(f^r;\eta) \to HM_*^w(f^{r'};\eta) \tag{6.3.1}$$

agrees with

$$(-1)^{\frac{(n+w)(n+w-1)}{2}} \underbrace{\iota \circ \cdots \circ \iota}_{r'-r}.$$

The proof of Theorem 6.0.1 is now basically just a corollary.

Proof of Thm. 6.0.1. Since continuation and inclusion maps both preserve the decomposition according to orientability, the colimits also split:

$$SH^*(T^*M;\mathbb{Z}) = SH_0^*(T^*M;\mathbb{Z}) \oplus SH_{-1}^*(T^*M;\mathbb{Z})$$
$$H_*(\mathcal{L}M;\eta) = H_*^0(\mathcal{L}M;\eta) \oplus H_*^{-1}(\mathcal{L}M;\eta).$$

Equation (6.3.1) implies that

$$\mathcal{V}_{r'} \circ \mathcal{F}^r : HM_*^w(f^r;\eta) \to HM_*^w(f^{r'};\eta)$$

agrees with $\iota^{r'-r}$ up to a sign, and thus taking the colimit on each orientation component yields that $\mathcal{V} \circ \mathcal{F}$ is an isomorphism. □

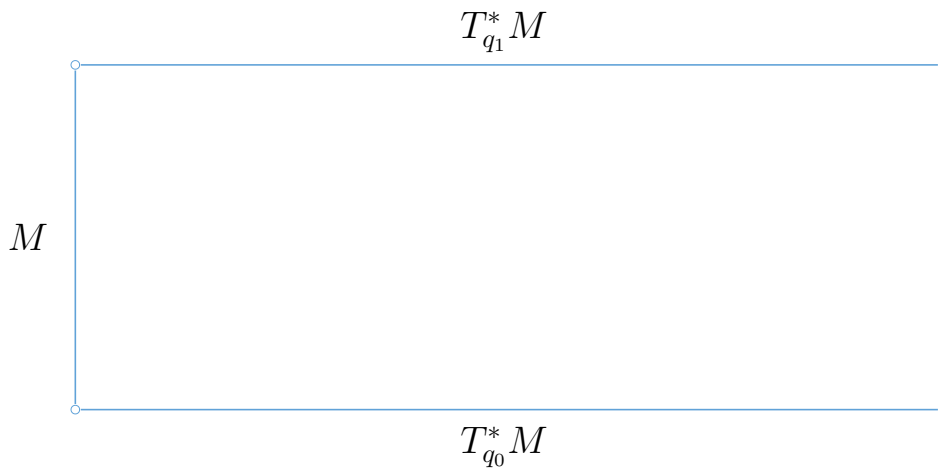
6.3.a) Two Cotangent Fibres and the Zero Section

Along the rest of the proof of Theorem 6.0.1, we are going to introduce a plethora of new spaces that we will need for cobordism arguments. However, we are going to make sure to illustrate the usefulness of these spaces at each step.

Consider the punctured positive half-strip

$$T^+ := [0,+\infty) \times [0,1] \setminus \{(0,0),(0,1)\} \tag{6.3.2}$$

equipped with coordinates (s,t) and complex structures $j\partial_s = \partial_t$.



Choose a function $[0,1] \to [0,1]$ which is the identity on the boundary and is in addition locally constant near the boundary. Define

$$\tau^+ : T^+ \to [0,1]$$

to be the composition of this function and the projection $(s,t) \mapsto t$.

We now want to define pseudo-holomorphic maps from T^+ into T^*M. Given q_0 and q_1 in M, let $\mathcal{R}(T^*_{q_0}M, M, T^*_{q_1}M)$ be the space of finite energy maps

$$u : T^+ \to T^*M$$

which solve the differential equation

$$(Du - d\tau^+ \otimes X_h)^{0,1} = 0 \tag{6.3.3}$$

with respect to an s-independent complex structure J_t on T^*M. In order to use the integrated maximum principle, we impose certain boundary conditions:

$$\begin{aligned} u((0,+\infty) \times \{1\}) &\subset T^*_{q_1}M \\ u(\{0\} \times (0,1)) &\subset M \\ u((0,+\infty) \times \{0\}) &\subset T^*_{q_0}M \end{aligned} \tag{6.3.4}$$

We have the following result:

Lemma 6.3.2. *Let q be an element of M. The constant map is the unique element of the moduli space $\mathcal{R}(T^*_qM, M, T^*_qM)$.*

Proof. First, we are going to translate the situation into the framework of Section 6.1.e). Define $\Sigma := T^+$, the boundary then decomposes as

$$\begin{aligned} \partial\Sigma &= \partial^n\Sigma \cup \partial^l\Sigma \quad \text{with} \\ \partial^n\Sigma &= \{0\} \times [0,1] \\ \partial^l\Sigma &= ((0,+\infty) \times \{0\}) \cup ((0,+\infty) \times \{1\}). \end{aligned}$$

An argument analagous to that of Lemma 6.1.13 then shows that for any $u \in \mathcal{R}(T_q^*M, M, T_q^*M)$, there is a chord $x \in \mathfrak{X}_h(T_q^*M, T_q^*M)$ which is the limit of u for $s \to +\infty$.

Lemma 6.1.8 states that the elements of $\mathfrak{X}_h(T_q^*M, T_q^*M)$ contained in D_2^*M are in a bijective correspondence with geodesics between q and q of length bounded by 2. The only geodesic that can ever connect q to itself is the constant map, thus $\mathfrak{X}_h(T_q^*M, T_q^*M)$ consists of a single element. Since X_h vanishes along the zero section, the constant chord $x(t) = 0 \in T_q^*M$ is the unique element of $\mathfrak{X}_h(T_q^*M, T_q^*M)$.

Next, we want to compute the topological energy of u. For this, we first compute the action of the unique chord x linked to u:

$$\begin{aligned}\mathcal{A}(x) &= \int_0^1 h(x(t))dt - \int x^*(\lambda) \\ &= \int_0^1 h(x(t))dt - \int_0^1 \lambda(\dot{x}(t))dt \\ &= 0,\end{aligned}$$

since $h(x(t)) = 0$ as $\rho(x(t)) = 0$ and $\dot{x}(t) = 0$ because $x(t)$ is constant. By Equation (6.1.16), the topological energy is given by:

$$\begin{aligned}E_{\text{top}}(u) &= \int_{\partial^n\Sigma} u^*(\lambda) - \int_{\partial^n\Sigma} u^*(h)d\tau^+ + \underbrace{\mathcal{A}(x)}_{=0} \\ &= \int_0^1 \lambda\Big(Du(0,t)\Big[\frac{\partial}{\partial t}\Big]\Big)dt + \int_0^1 h(u(0,t))\frac{\partial\tau^+}{\partial t}dt \\ &\overset{(1)}{=} \int_0^1 \lambda\Big(Du(0,t)\Big[\frac{\partial}{\partial t}\Big]\Big)dt \\ &\overset{(2)}{=} 0,\end{aligned}$$

where (1) follows from the fact that h vanishes on M and (2) since by Lemma 6.1.3, we have that $\lambda_{|_M} = 0$. The geometric energy then equals

$$\begin{aligned}E_{\text{geo}}(u) &= E_{\text{top}}(u) + \int_\Sigma u^*(h)d^2\tau^+ \\ &= 0 + 0,\end{aligned}$$

by Lemma 6.1.15. Lemma 6.1.12 then implies that the image of u is contained in an orbit of X_0; implying that the constant map is the unique element of $\mathcal{R}(T_q^*M, M, T_q^*M)$. □

This lemma can be generalised to the case where q_0 and q_1 differ, but are close enough together. Introduce the gauge transformation

$$\tilde{u}(s,t) := \Phi^{-\tau^+(t)}(u(s,t)),$$

where Φ^t is the time-t Hamiltonian flow of h.

Note that $\tilde{u}$ then maps $(T^*_{q_0}M, M, T^*_{q_1}M)$ to $(T^*_{q_0}M, M, \Phi^{-1}(T^*_{q_1}M))$, and thus for q_0 and q_1 close enough, we can use Lemma 6.3.2 to show that there is a unique element in $\mathcal{R}(T^*_{q_0}M, M, T^*_{q_1}M)$. A regularity analysis similar to the one of [Abo14, Lem. 5.4.7.] then also shows that the consant map is regular.

Let us now consider the union over all punctures of T^+:

$$\mathcal{R}(\mathcal{L}^2M) := \bigsqcup_{(q_0,q_1)\in\mathcal{L}^2M} \mathcal{R}(T^*_{q_0}M, M, T^*_{q_1}M). \tag{6.3.5}$$

One can once again topologise this space by seeing it as a parametrised total space of the bundle

$$C^\infty(T^+, T^*M; \mathcal{L}^2M) \to \mathcal{L}^2M$$

whose fibre at a point (q_0, q_1) is the space

$$C^\infty(T^+, T^*M; (T^*_{q_0}M, M, T^*_{q_1}M))$$

of smooth maps from T^+ to T^*M which decay exponentially to the unqiue chord connecting q_0 to q_1 at ∞, and to the intersection point between $T^*_{q_i}M$ and M at $(0, i)$ for $i \in \{0, 1\}$ and thus have boundary conditions given by Equation (6.3.4).

Next, we identify a subset of $\mathcal{R}(\mathcal{L}^2M)$ that projects diffeomorphically onto a subset of $\mathcal{L}^2M$. In order to achieve this, we introduce two Banach bundles. The first one is the space of $W^{1,p}$-sections of the pullback of T^*M to T^+, which map the boundary to T^+ to those in Equation (6.3.4):

$$W^{1,p}((T^+, \partial T^+); (u^*(TT^*M), (u^*(T^*_{q_0}M), u^*(TM), u^*(TT^*_{q_1}M)))).$$

The second bundle is the space of L^p-sections of $u^*(TT^*M)$:

$$L^p(T^+, u^*(T^*M)).$$

Linearising the Cauchy-Riemann operator and taking into account the parametrisation of $\mathcal{R}(\mathcal{L}^2M)$ by $\mathcal{L}^2M$, we obtain a map

$$\begin{aligned} W^{1,p}((T^+, \partial T^+); (u^*(TT^*M), (u^*(T^*_{q_0}M), u^*(TM), u^*(TT^*_{q_1}M)))) \oplus T\mathcal{L}^2M \\ \to L^p(T^+, u^*(T^*M)). \end{aligned} \tag{6.3.6}$$

Lemma 6.3.3. *The restriction of the projection to $\mathcal{L}^2M$ is a local diffeomorphism in a neighbourhood of the constant triangles.*

Proof (Sketch). We will follow the discussion above [Abo14, Lem. 5.5.6.]. Whenever the map (6.3.6) is surjective at an element u of $\mathcal{R}(\mathcal{L}^2M)$, its kernel consists of the tangent space $T_u(\mathcal{R}(\mathcal{L}^2M))$. Since being regular is an open condition for solutions to Cauchy-Riemann operators, we have that there is an open neighbourhood in $\mathcal{R}(\mathcal{L}^2M)$ of the constant triangles which consists of regular elements, hence which is a manifold of dimension $2n$. In addition, for a constant triangle, the restriction of the map (6.3.6) to the first factor is an isomorphism; implying that the kernel of the operator (6.3.6) projects isomorphically onto the second factor. This

shows that the tangent space of the moduli space in a neighbourhood of constant points projects isomorphically to the tangent space of $\mathcal{L}^2 M$ under ev.

The restriction of the projection to $\mathcal{L}^2 M$ is therefore a local diffeomorphism in a neighbourhood of constant triangles, as the set of constant triangles is a submanifold of $\mathcal{R}(\mathcal{L}^2 M)$, on which the projection map is injective. We conclude that the projection to $\mathcal{L}^2 M$ is a diffeomorphism from a neighbourhood of the set of constant triangles to a neighbourhood of the diagonal. □

The above description of this neighbourhood appears a little elusive. To remedy this, introduce the subset

$$\mathcal{R}(\mathcal{L}^2_\delta M) \subset \mathcal{R}(\mathcal{L}^2 M)$$

corresponding to points whose distance is bounded by a constant δ. We then have:

LEMMA 6.3.4 ([Abo14, Lem. 5.5.6.]). *If δ is sufficiently small, $\mathcal{R}(\mathcal{L}^2_{2\delta} M)$ is a smooth manifold of dimension $2n$, and the projection map to $\mathcal{L}^2_{2\delta} M$ is a diffeomorphism.*

Proof (Sketch). It suffices to show that we can choose δ small such that the entirety of $\mathcal{R}(\mathcal{L}^2_{2\delta} M)$ is contained in the neighbourhood described above. Using the integrated maximum principle, one can show the analogue of Lemma 6.2.3, i.e. that the image of all elements of $\mathcal{R}(\mathcal{L}^2 M)$ are contained in $D^* M$. The Gromov-Floer compactification is then able to produce a compactification of $\mathcal{R}(T^*_{q_0} M, M, T^*_{q_1} M)$ by considering disc bubbling, sphere bubbling and breakings of Floer trajectories at the corner of the triangle. All of these possible phenomena can be discarded: the first two by exactness of $T^* M$ (and of the Lagrangians M and $T_{q_i} M$), and the last by the fact that $M \cap T^*_{q_i} M$ is the single point $0 \in T^*_{q_i} M$, and that there is a unique chord of h connecting $T^*_{q_0} M$ and $T^*_{q_1} M$. We conclude that the compactification is just the space itself, thus the projection is proper and for δ small enough, all elements of $\mathcal{R}(\mathcal{L}^2_{2\delta} M)$ are close to the constant maps lying over the diagonal. □

The space $\mathcal{R}(\mathcal{L}^2_{2\delta} M)$ is equipped with an evaluation map to the space of paths between points whose distance is smaller than 2δ. The map is given by

$$\begin{aligned} \mathcal{R}(\mathcal{L}^2_{2\delta} M) \to \mathcal{P}(\mathcal{L}^2_{2\delta} M) &:= \{\delta : [0,1] \to M \mid d(\gamma(0), \gamma(1)) \leq 2\delta\} \\ u &\mapsto u|_{\{0\}\times[0,1]}. \end{aligned} \tag{6.3.7}$$

In addition, we may assume the image of every element of $\mathcal{R}(\mathcal{L}^2_{2\delta} M)$ lies in a geodesically convex ball by maybe shrinking δ further, implying:

LEMMA 6.3.5 ([Abo14, Lem. 5.5.7.]). *If δ is sufficiently small, the composition*

$$\mathcal{L}^2 M \xrightarrow{\sim} \mathcal{R}(\mathcal{L}^2_{2\delta} M) \longrightarrow \mathcal{P}(\mathcal{L}^2_{2\delta} M)$$

is homotopic to the map $\mathcal{L}^2 M \to \mathcal{P}(\mathcal{L}^2_{2\delta} M)$ which assigns to two points the unique geodesic between them.

6.3.b) Moduli Space of Degenerate Annuli

In this section, we are going to construct a cobordism between two moduli spaces that will show that the composition $\mathcal{V} \circ \mathcal{F}$ is the identity on loop homology (possibly up to a sign).

We start by constructing a moduli space $\mathcal{A}_r^0$ that induces, in the limit over r, the identity on loop homology. Assign to each element $\vec{q} \in \mathcal{L}^r M$ the moduli space

$$\mathcal{A}_r^0(\vec{q}) := \mathcal{R}(T^*_{q_0}M, M, T^*_{q_1}M) \times \cdots \times \mathcal{R}(T^*_{q_{r-1}}M, M, T^*_{q_0}M), \tag{6.3.8}$$

whose elements form degenerate annuli by gluing the triangles along their common corners.

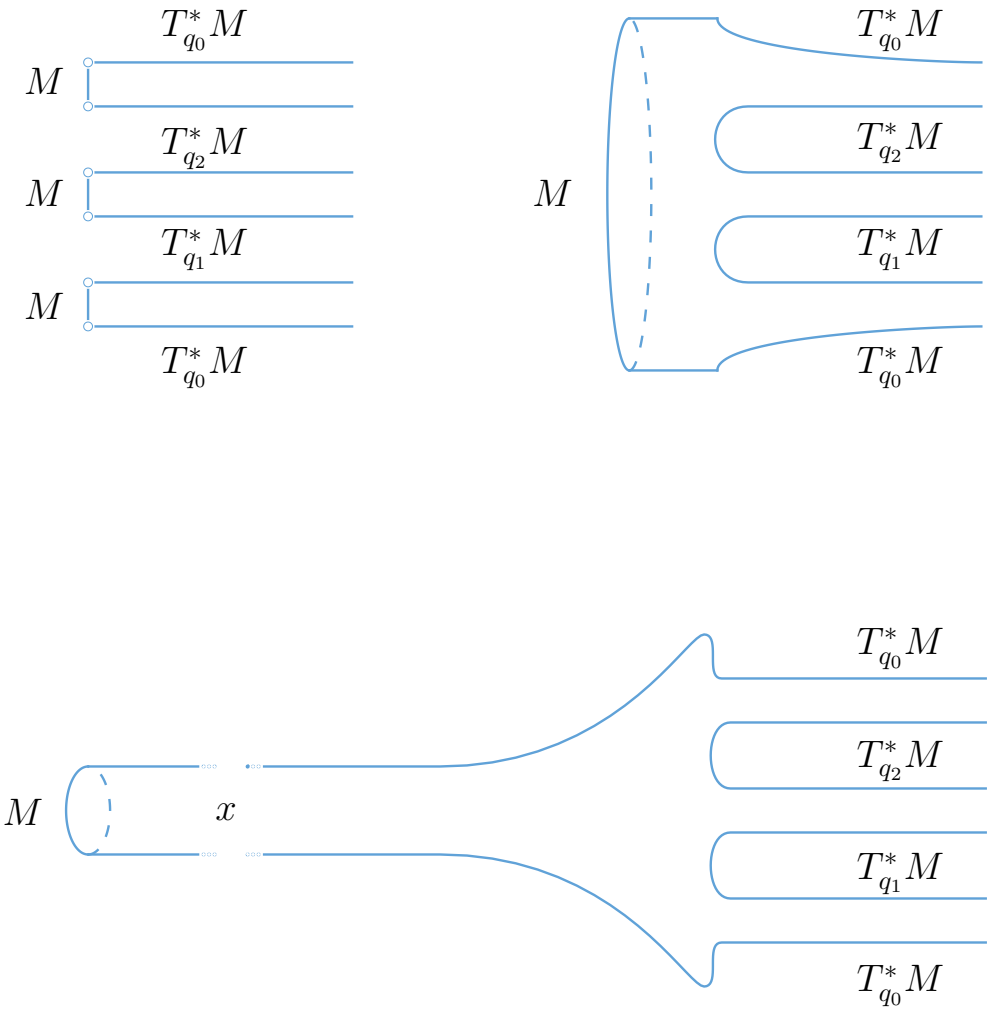


Figure I.6: Gluing of holomorphic triangles.

By taking the union over all $\vec{q} \in \mathcal{L}^r M$, we obtain the parametrised moduli space

$$\mathcal{A}_r^0 := \bigsqcup_{\vec{q} \in \mathcal{L}^r M} \mathcal{A}_r^0(\vec{q}). \tag{6.3.9}$$

By choosing δ sufficiently small such that Lemma 6.3.4 and Lemma 6.3.5 hold, and assuming that

$$\delta \leq \delta_r^i \leq 2\delta, \tag{6.3.10}$$

Lemma 6.3.4 implies

LEMMA 6.3.6 ([Abo14, Lem. 5.5.8.]). *$\mathcal{A}_r^0(\mathcal{L}^r M)$ is a smooth manifold of dimension $r - n$, and the projection to $\mathcal{L}^r M$ is a diffeomorphism.*

REMARK 6.3.7. This lemma shows that the uniqueness of chords connecting cotangent spaces with close basepoints has profound consequences for the space $\mathcal{A}_r^0(\mathcal{L}^r M)$; it is entirely defined by the basepoints of the glued cotangent spaces.

By concatenating the boundary components that map into M, we obtain an evaluation map. In particular, identifying the intervals $\left[\frac{i}{r}, \frac{i+1}{r}\right]$ with these segments, we obtain

$$\begin{aligned} &\mathrm{ev} : \mathcal{A}_r^0(\mathcal{L}^r M) \to \mathcal{L}M \\ &\mathrm{ev}(v_0, \ldots, v_r)(t) := v_i(0, rt - i) \quad \text{if } t \in \left[\frac{i}{r}, \frac{i+1}{r}\right]. \end{aligned} \tag{6.3.11}$$

Since we chose δ small, we are able to infer the following result from Lemma 6.3.5:

LEMMA 6.3.8 ([Abo14, Lem. 5.5.10.]). *Assuming that Inequality (6.3.10) holds, we have that the composition*

$$\mathcal{L}^r M \xrightarrow{\sim} \mathcal{A}_r^0(\mathcal{L}^r M) \longrightarrow \mathcal{L}M$$

is homotopic to the natural map $\mathcal{L}^r M \to \mathcal{L}M$.

The other end of the moduli space, that will later on be shown to control $\mathcal{V} \circ \mathcal{F}$, is defined as follows: given a Hamiltonian H^- of slope larger than $2r$, define

$$\mathcal{A}_r^\infty(\vec{q}) := \bigsqcup_{x \in \mathcal{O}(H^-)} \mathcal{M}(x) \times \mathcal{M}(x, \vec{q}), \tag{6.3.12}$$

which we can think of as the collection of infinitely long annuli, see the bottom drawing of Figure I.6. We also parametrise this space:

$$\mathcal{A}_r^\infty(\mathcal{L}^r M) := \bigsqcup_{x \in \mathcal{O}(H^-)} \mathcal{M}(x) \times \mathcal{M}(x, \mathcal{L}^r M).$$

The results of Lemma 5.2.2 and Lemma 6.2.5 immediately show that $\mathcal{A}_r^\infty(\mathcal{L}^r M)$ is a smooth manifold of dimension nr.

6.3.c) Cobordism Between Degenerate Annuli

Consider the moduli space of Riemannian surfaces biholomorphic to the annulus

$$\Sigma^R := [0, R] \times S^1$$

equipped with coordinates (s, t). By removing the points $z_i = (R, \frac{i}{r})$ from the boundary, we obtain the punctured Riemannian surface

$$\Sigma_r^R := \Sigma^R \setminus \{z_i\}_{i=1}^r.$$

The boundary of Σ_r^R has $r + 1$ boundary components: we denote by $\partial^0 \Sigma_r^R$ the boundary component corresponding to setting the first coordinate equal to 0, and by $\{\partial^i \Sigma_r^R\}_{i=1}^r$ the segment

$$s = R, \quad t \in \Big[\frac{i-1}{r}, \frac{i}{r}\Big].$$

There is a natural compactification of this moduli space, which we simply denote by $\mathcal{A}_r$, corresponding to allowing $R = \{0, +\infty\}$; these degenerate annuli are

$$\Sigma_r^\infty := Z^+ \sqcup Z_r^- \tag{6.3.13}$$

$$\Sigma_r^0 := \underbrace{T^+ \sqcup T^+ \sqcup \cdots \sqcup T^+}_{r-\text{times}}, \tag{6.3.14}$$

where Z_r^- is the negative punctured cylinder,

$$Z^+ = [0, +\infty) \times S^1 \text{ and } T^+ = ([0, 1] \times \mathbb{R}) \setminus \{(0, 0), (1, 0)\}.$$

The topology on the boundary $\mathcal{A}_r$ arises from gluing as follows: given $S \geq 0$ together with the half-cylinders Z^+ and Z^-, we can glue them by the a biholomorphic map

$$\Big(\frac{2}{3}S, +\infty\Big) \times S^1 \subset Z^+ \to \Big(-\infty, -\frac{2}{3}S\Big) \times S^1 \subset Z^-$$
$$(a, t) \mapsto (f(a), t),$$

where f is a monotonically increasing smooth function

$$f : \Big(\frac{2}{3}S, +\infty\Big) \to \Big(-\infty, -\frac{2}{3}\Big) \text{ such that}$$

- f is monotonically increasing,
- $\lim_{a \to \frac{2}{3}S} f(a) = -\infty$,
- $f(S) = -S$,
- $\lim_{a \to +\infty} f(a) = -\frac{2}{3}S$.

Then $(\frac{2}{3}S, S] \subset Z^+$ gets identified with $(-\infty, -S] \subset Z^-$ and $[S, +\infty) \subset Z^+$ with $[-S, -\frac{2S}{3})$.

Moreover, this construction yields a biholomorphic map to the closed cylinder $[0, 2S] \times S^1$, which after puncturing the negative end is just Σ^{2S}.

By identifying T^+ with $(\mathbb{R} \times [0, 1]) \setminus \{(0, 1)\}$, and gluing r-copies of T^+ by a similar gluing construction as above, one can see that $T^+ \#_S \ldots \#_S T^+$ is biholomorphic to $\Sigma^{\frac{1}{S}}$.

Next, we give a description of the Floer data on $\mathcal{A}_r$ and we will follow the discussion of [Abo14, Sec. 5.5.3.]. Since Σ_r^0 is the disjoint union of r copies of T^+, Equation (6.3.3) defines a pseudo-holomorphic curve equation on every component. Moreover, Equation (5.2.2) and (6.2.5) define pseudo-holomorphic equation on the components of Σ_r^∞. We now want to extend these to other points of the moduli space $\mathcal{A}_r$. Recall that we have fixed positive strip-like ends near the boundary punctures of Σ_r^0 and Σ_r^∞ respectively. By gluing, we obtain strip-like ends on the nearby elements of $\mathcal{A}_r$, and we extend these choices to strip-like ends

$$\varepsilon_i^R : [0, +\infty) \times [0, 1] \to \Sigma_r^R, \quad 1 \leq i \leq r,$$

converging to the puncture z_i.

The pseudo-holomorphic equation we impose on maps $u : \Sigma_r^R \to T^*M$ will look like

$$(Du - \alpha^R \otimes X_{H^R})^{0,1} = 0, \tag{6.3.15}$$

with

1. α^R: a 1-form,
2. H_z^R: a family of linear Hamiltonians parametrised by $z \in \Sigma_r^R$,
3. J_z^R: a family of almost complex structures parametrised by $z \in \Sigma_r^R$ that is in addition convex near S_2^*M.

Furthermore, we impose the following technical assumptions:

(6.3.16.i) α^R is closed, its restriction to the boundary components $\{\partial^i \Sigma_r^R\}_{i=1}^r$ vanishes, and there are positive strip-like ends at the punctures z_i the pull-back of α^R agrees with $b_i^R dt$ for real number $b_i^R \in (\delta_i^r, 2)$.

(6.3.16.ii) The Hamiltonian H_z^R agrees with h if z lies in a neighbourhood of the punctures $\{z_i\}_{i=1}^r$. If z lies on $\partial^0 \Sigma_r^R$, we assume that the Hamiltonian flow of $X_{H_z^R}$ vanishes on M.

(6.3.16.iii) There exists a real number b^R such that $2\sum_{i=1}^r b_i^R \leq b^R$, and a smooth function f^R on Σ^R, which vanishes near the punctures such that

$$df^R \wedge \alpha^R \leq 0.$$

(6.3.16.iv) $H^R|_{T^*M\setminus D_2^*M} = 2\rho - 2 + f^R\Big(\Big(\frac{b^R}{\sum_{i=1}^r b_i^R} - 2\Big)\rho + 2\Big)$.

We now explain how to produce pseudo-holomorphic curve equations on the surfaces constructed by the gluing constructions. For the surface Σ_r^∞, we have assumed that the 1-form α^- on Z^- agrees with dt near the negative puncture, while Z^+ carries the 1-form dt in the discussion of Section 5.2. Since the restrictions of these forms to the glued regions agree, we conclude that, for S large enough, the surface Σ^{2S} naturally carries a 1-form that we denote α^{2S}. Similarly, the family of functions H^- on Z^- is assumed to agree with a fixed time-dependent Hamiltonian H_t along the end, as does the family of Hamiltonians on Z^+ fixed in Equation (5.2.1). We therefore obtain a family of Hamiltonians on Σ^{2S} that we denote H^{2S}.

We repeat the same procedure to produce data near the surface Σ_r^0: first, we set

$$H_z^R = h$$

whenever R is sufficiently close to 0. Moreover, the 1-form $d\tau$ on the surface T^+ vanishes by assumption near the punctures (0,0) and (0,1). There is therefore a natural 1-form on the result of gluing r copies of this surface along these ends, which will be the 1-form α^R. We now choose data α^R, H^R and J^R varying smoothly for $R \in (0, +\infty)$ which, when restricted to a neighbourhood of 0 and ∞ is obtained by the gluing construction, and which satisfy Conditions (6.3.16.i) - (6.3.16.iv); to see that this can indeed be achieved, the key observation is that the space of data satisfying these properties is contractible, so there is no obstruction to extending the choice from neighbourhoods of $R = \{0, +\infty\}$ to arbitrary R.

Definition 6.3.9. *Given such data, and a point $\vec{q} \in \mathcal{L}^r M$, we define $\mathcal{A}_r^R(\vec{q})$ to be the space of finite energy maps*

$$w : \Sigma_r^R \to T^*M$$

which map $\partial^0\Sigma$ to M and $\partial^i\Sigma$ to $T_{q_i}M$ and solve Equations (6.3.15).

The next result follows in exactly the same manner as Lemma 6.2.3.

Lemma 6.3.10. *The moduli space $\mathcal{A}_r^R(\vec{q})$ is compact if $R \neq \infty$.*

Proof. This is again an application of the integrated maximum principle. Let u be an element of $\mathcal{M}(x, \vec{q})$, and assume by contradiction that its image is not entirely contained in D_2^*M. In particular, the inverse image $\Sigma := u^{-1}(T^*M \setminus D_l^*M)$ is not empty for l sufficiently close to 2. In addition, we choose l greater than 2 so that u is transverse to S_l^*M, and Σ is not empty.

By Equation ((6.3.16.iv)), the restriction of Equation (6.3.15) to Σ is

$$Du - \Big(2 + \Big(\frac{b^R}{\sum b_i} - 2\Big) f^R\Big)\alpha^R \otimes X_\rho\Big)^{0,1},$$

where the constant terms in ((6.3.16.iv)) disappear because the associated Hamiltonian flow vanishes. Most of the assumptions of Proposition 6.1.14 are now given

by construction; all that is left is to note that $\frac{b^R}{\sum b_i} - 2$ is positive by (6.3.16.iii) and thus it suffices to show that $d(f^R\alpha^R) \leq 0$, which is immediate by assumption (6.3.16.iii). We conclude from Proposition 6.1.14 that Σ is empty, contradicting the assumption. Since the constant l can be chosen arbitrarily close to 2, we have that $u(\Sigma_r^R) \subset D_2^*M$. □

6.3.d) Evaluation Maps

Taking the union over all parameters R and points $\vec{q}$, we obtain a moduli space of annuli:

$$\mathcal{A}_r(\mathcal{L}^r M) := \bigcup_{\vec{q}\in\mathcal{L}^r M} \bigcup_{R\in[0,+\infty]} \mathcal{A}_r^R(\vec{q}). \tag{6.3.17}$$

Since we chose the Floer data to vary smoothly, for generic data $\mathcal{A}_r(\mathcal{L}^r M)$ is a smooth manifold with boundary

$$\bigcup_{\vec{q}\in\mathcal{L}^r M} \mathcal{A}_r^0(\vec{q}) \text{ and } \bigcup_{\vec{q}\in\mathcal{L}^r M} \mathcal{A}_r^\infty(\vec{q}).$$

Using the identification of (6.3.13) and (6.3.14), we see that these equal $\mathcal{A}_r^0(\mathcal{L}^r M)$ and $\mathcal{A}_r^\infty(\mathcal{L}^r M)$ respectively, and it thus provides a cobordism between them. Since $\mathcal{A}_r^R(\mathcal{L}^r M)$ is compact for $R \neq \infty$, we set

$$\overline{\mathcal{A}_r^R}(\mathcal{L}^r M) := \mathcal{A}_r^R(\mathcal{L}^r M).$$

For $R = \infty$, the moduli space breaks up, and thus we instead have to use Gromov-Floer compactification:

$$\overline{\mathcal{A}_r^\infty}(\mathcal{L}^r M) := \bigcup_{x\in\mathcal{O}(H^-)} \overline{M}(x) \times \mathcal{M}(x, \vec{q})).$$

The compactification of $\mathcal{A}_r(\mathcal{L}^r M)$ is defined to be:

$$\overline{\mathcal{A}}_r(\mathcal{L}^r M) := \bigcup_{R\in[0,+\infty]} \overline{\mathcal{A}}_r^R(\mathcal{L}^r M),$$

equipped with the Gromov topology. Consider the evalutation map

$$\mathrm{ev} : \overline{\mathcal{A}}_r(\mathcal{L}^r M) \to \mathcal{L}M \tag{6.3.18}$$

which si defined for $R = 0$ in Equation (6.3.11) and for other values of R by restricting to $\partial^0\Sigma^R$, since by Definition 6.3.9 we have the inclusion $w(\partial^0\Sigma) \subset M$ for every solution w.

Since $\overline{\mathcal{A}}_r$ is compact, its image under ev is also compact and thus we can find a uniform bound on the C^1-norm of the curves $\mathrm{ev}(u)$. As in the proof of Lemma 5.2.4, we may thus choose $r' > 0$ large enough such that for each $w \in \overline{\mathcal{A}}_r(\mathcal{L}^r M)$ and each interval $I \subset S^1$ of length less than $\frac{1}{r'}$, the length in M of $\mathrm{ev}(w)(I)$ is less than δ, where δ is chosen small enough such that the conclusions of Lemma 6.3.4 and 6.3.5 hold. This yields an evaluation map:

$$\begin{aligned} \mathrm{ev}_{r'} : \overline{\mathcal{A}}_r(\mathcal{L}^r M) &\to \mathcal{L}^{r'} \\ w &\mapsto \left(\mathrm{ev}(w)(0), \mathrm{ev}(w)\left(\frac{1}{r'}\right), \ldots, \mathrm{ev}(w)\left(\frac{r'-1}{r'}\right)\right), \end{aligned} \tag{6.3.19}$$

with the property that the following diagram commutes up to homotopy

$$\begin{array}{ccc} \overline{\mathcal{A}}_r & \xrightarrow{\mathrm{ev}_{r'}} & \mathcal{L}^r M \\ & \searrow^{\mathrm{ev}} & \downarrow \\ & & \mathcal{L}M \end{array}$$

For $R = \infty$, recall that $\mathcal{A}_r^\infty(\mathcal{L}^r M) = \bigsqcup_{x \in \mathcal{O}(H')} \mathcal{M}(x) \times \mathcal{M}(x, \vec{q})$ and that $\mathrm{ev}_{r'}$ samples the image of $\partial^0 \Sigma^R$ under $u \in \mathcal{A}_r^\infty(\mathcal{L}^r M)$ at r' points equally spaced in time. Since $\partial^0 \Sigma^R$ is entirelly contained in the $\mathcal{M}(x)$ component, $\mathrm{ev}_{r'}|_{\mathcal{A}_r^\infty(\mathcal{L}^r M)}$ is the same as the composition

$$\bigsqcup_{x \in \mathcal{O}(H^-)} \mathcal{M}(x) \times \mathcal{M}(x, \vec{q}) \longrightarrow \bigsqcup_{x \in \mathcal{O}(H^-)} \mathcal{M}(x) \longrightarrow \mathcal{L}^{r'} M,$$

where the first map is simply the projection and the second map the evaluation map $\mathrm{ev}_{r'}$ of Equation (5.2.6).

Lemma 6.3.11. *The restriction of* $\mathrm{ev}_{r'}$ *to* $\mathcal{A}_r^0(\mathcal{L}^r M)$ *is homotopic to the composition*

$$\mathcal{A}_r^0(\mathcal{L}^r M) \xrightarrow{\pi_{\vec{q}}} \mathcal{L}^r M \xrightarrow{\iota} \mathcal{L}^{r+1} M \xrightarrow{\iota} \dots \xrightarrow{\iota} \mathcal{L}^{r'} M,$$

where $\pi_{\vec{q}}$ *is the projection onto the basepoints of the cotangent spaces.*

Proof. Note that projection map $\pi_{\vec{q}}$ is actually just the evaluation map of Equation (6.3.11) at times $\left\{0, \frac{1}{r}, \dots, \frac{r-1}{r}\right\}$, thus given $w \in \overline{\mathcal{A}}_r(\mathcal{L}^r M)$, we have that

$$\underbrace{\iota \circ \dots \iota}_{r'-r} \circ \pi_{\vec{q}}(w) = \Big(\underbrace{\mathrm{ev}(w)(0), \dots, \mathrm{ev}(w)(0)}_{r'-r+1}, \mathrm{ev}(w)\Big(\frac{1}{r}\Big), \dots, \mathrm{ev}(w)\Big(\frac{r-1}{r}\Big)\Big),$$

and by definition

$$\mathrm{ev}_{r'}(w) = \Big(\mathrm{ev}(w)(0), \dots, \mathrm{ev}(w)\Big(\frac{r'-1}{r'}\Big)\Big).$$

Since all the points in the image of $\mathrm{ev}(w)$ are path-connected, we conclude. □

Next, we will briefly comment on the orientation of $\mathcal{A}_r(\mathcal{L}^r M)$. By construction, the moduli space of annuli admits a natural map

$$\pi_{\vec{q}} : \mathcal{A}_r(\mathcal{L}^r M) \to \mathcal{L}^r M$$

which is the projection to the parameter space of (6.3.17). As the exposition of [Abo14, Sec. 5.5.5.] shows, this map induces a relative orientation of $\mathcal{A}_r(\mathcal{L}^r M)$ by $\mathcal{L}^r M$ and there is a natural isomorphism

$$|\mathcal{A}_r(\mathcal{L}^r M)| \cong \pi_{\vec{q}}^* |\mathcal{L}^r M|. \tag{6.3.20}$$

Let $\mathcal{L}_0^r M$ and $\mathcal{L}_1^r M$ be the components of $\mathcal{L}^r M$ corresponding to piecewise geodesics along which M is or respectively is not orientable. We have the following result that will clarify the sign of $\iota^{r'-r}$ in Proposition 6.3.1.

Lemma 6.3.12 ([Abo14, Lem. 5.5.23.]). *The relative orientation of* $\mathcal{A}_r^0(\mathcal{L}_w^r M)$ *differs from the induced orientation as a boundary stratum of* $\mathcal{A}_r(\mathcal{L}_w^r M)$ *by a sign whose parity is*

$$\frac{(n+w)(n+w-1)}{2} + 1.$$

□ref.

6.3.e) Rigidifying Moduli Spaces

Note that we now have two maps

$$\pi_{\vec{q}} : \mathcal{A}_r(\mathcal{L}^r M) \to \mathcal{L}^r M \quad \text{, and} \tag{6.3.21}$$

$$\mathrm{ev}_{r'} : \mathcal{A}_r(\mathcal{L}^r M) \to \mathcal{L}^{r'} M. \tag{6.3.22}$$

The picture one should have in mind is the following: given a solution $w \in \mathcal{A}_r(\mathcal{L}^r M)$ we have the following picture

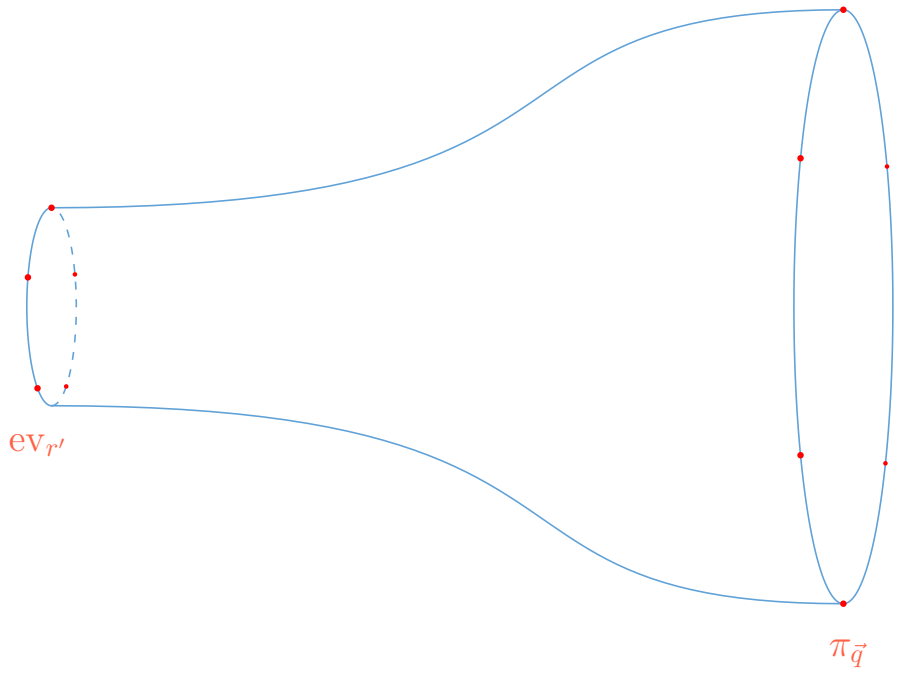


Figure I.7: The red dots are the images of the respective maps for the case $r' = 5$ and $r = 6$.

thus one should think of $\mathrm{ev}_{r'}$ as acting on the left and $\pi_{\vec{q}}$ as acting on the right end of w. Given a critical point y of f^r, y' of $f^{r'}$ we can thus consider the fibre product

$$W^s(y') \times_{\mathrm{ev}_{r'}} \times \mathcal{A}_r^+(\mathcal{L}^r M) \times_{\pi_{\vec{q}}} W^u(y).$$

If the Floer data on the annuli and the Morse functions f^r and $f^{r'}$ are chosen generically, this fibre product is transverse; implying that it is a manifold with boundary. The dimension of $\mathcal{A}_r^+(\mathcal{L}^r M) \times_{\pi_{\vec{q}}} W^u(y)$ is $\mathrm{ind}(y) + 1$, and given that the codimension of $W^s(y')$ is $\mathrm{ind}(y')$, we have

$$\dim(\mathcal{A}_r(y', y)) = \mathrm{ind}(y) - \mathrm{ind}(y') + 1.$$

Lemma 6.3.13. *There is a natural isomorphism*

$$|\mathcal{A}_r(y', y)| \otimes o_y \otimes |W^s(y')| \cong |\mathcal{L}^{r'} M|. \tag{6.3.23}$$

Proof. First, consider $\mathcal{A}_r^+(\mathcal{L}^r M) \times_\pi W^u(y)$. This fibre product induces the isomorphism of orientation lines:

$$|\mathcal{L}^r M| \otimes |\mathcal{A}_r^+(\mathcal{L}^r M) \times_\pi W^u(y)| \cong |\mathcal{A}_r^+(\mathcal{L}^r M)| \otimes \underbrace{|W^u(y)|}_{=o_y}$$

Taking the other fibre product $W^s(y') \times_{\mathrm{ev}_{r'}} (\mathcal{A}_r^+(\mathcal{L}^r M) \times_\pi W^u(y))$ yields:

$$|\mathcal{L}^{r'} M| \otimes |\mathcal{A}_r(y', y)| \cong |W^s(y')| \otimes |\mathcal{A}_r^+(\mathcal{L}^r M) \times_\pi W^u(y)|.$$

Combining the two, we obtain

$$|\mathcal{L}^{r'} M| \otimes |\mathcal{A}_r(y', y)| \cong |W^s(y')| \otimes o_y \otimes |\mathcal{A}_r^+(\mathcal{L}^r M)| \otimes |\mathcal{L}^r M|^{-1}.$$

Rearranging and using the natural isomorphism $l \cong l^{-1}$ of line bundles concludes the proof. □

We introduce the standard compactification

$$\overline{\mathcal{A}_r}(y', y) := \overline{W^s}(y') \times_{\mathrm{ev}_{r'}} \overline{\mathcal{A}_r}(\mathcal{L}^r M) \times_\pi \overline{W^u}(y).$$

The following result is an immediate consequence of the dimensions of the spaces involved to define $\mathcal{A}_r(y', y)$ and of the implicit function theorem.

LEMMA 6.3.14 ([Abo14, lem. 5.5.28.]). *For generic Floer data, $\overline{A}_r(y', y)$ is empty whenever*

$$\mathrm{ind}(y) - \mathrm{ind}(y') + 1 < 0,$$

and it is a manifold with boundary whenever $\mathrm{ind}(y) - \mathrm{ind}(y') + 1$ *equals 0 or 1.*

□ref.

6.3.f) Definition of the Maps ι^0 and ι^∞

Let $\mathcal{A}_r^R(y', y)$ be the fibre product

$$\mathcal{A}_r^R(y', y) := W^s(y') \times_{\mathrm{ev}_{r'}} \mathcal{A}_r^R(\mathcal{L}^r M) \times_\pi W^u(y).$$

Putting Equation (4.3.1) and (6.3.20) together yields the isomorphism

$$\mathcal{A}_r^R(y', y)| \otimes o_y \cong o_{y'}.$$

Assume now that $\mathrm{ind}(y) = \mathrm{ind}(y')$, so that the $\mathcal{A}_r^R(y', y)$ are 0-dimensional if they are regular. Since $\mathcal{A}_r^R(y', y)$ is now assumed to be regular, each $w \in \mathcal{A}_r^R(y', y)$ together with flow lines $\gamma' \in W^s(y')$ and $\gamma \in W^u(y)$ satisfying the necessary limit conditions induce a map

$$o_y \to o_{y'}.$$

The annulus w also induces a map on the local system η on $\mathcal{L}M$, thus also on the restrictions to $\mathcal{L}^r M$ and $\mathcal{L}^{r'} M$. Composing this map with the flow lines γ and γ' defines a map

$$\eta_y \to \eta_{y'}.$$

Tensoring these two maps yields:

$$\iota_{(\gamma',w,\gamma)} : o_y \otimes \eta_y \to o_{y'} \otimes \eta_{y'}.$$

Taking the direct sum over all of these maps, we obtain:

$$\begin{aligned} \iota^R : CM_*(f^r;\eta) &\to CM_*(f^r;\eta) \\ \iota^R|_{o_y\otimes\eta_y} &\equiv \bigoplus_{\mathrm{ind}(y')=\mathrm{ind}(y)} \sum_{(\gamma',w,\gamma)\in\mathcal{A}_r^R(y',y)} \iota_{(\gamma',w,\gamma)}. \end{aligned} \tag{6.3.24}$$

Lemma 6.3.15. *The map ι^R of Equation (6.3.24) is a chain map, provided that the involved moduli spaces $\mathcal{A}_r^R(y',y)$ are regular.*

Proof (Sketch). The proof follows the same mold as the proof of Lemma 5.2.6 and thus we will only provide the cobordism. Given two critical points y' of $f^{r'}$ and y of f^r with $\mathrm{ind}(y) - \mathrm{ind}(y') = 1$. By a slight variation of Lemma 6.3.14, we have that $\overline{\mathcal{A}_r^R}(y',y)$ is a one dimensional manifold with boundary given by

$$\bigsqcup_{\mathrm{ind}(y_1')=\mathrm{ind}(y')+1} \mathcal{T}(y',y_1') \times \mathcal{A}_r^R(y_1',y) \quad \text{and}$$
$$\bigsqcup_{\mathrm{ind}(y)=\mathrm{ind}(y_0)+1} \mathcal{A}_r^R(y,y_0) \times \mathcal{T}(y_0,y).$$

The first component can be identified with $\partial \circ \iota^R$ and the second one with $\iota^R \circ \partial$. A careful analysis of Koszul signs, in addition to the fact that the sum of the maps associated to $\partial\overline{\mathcal{A}_r^R}(y',y)$ vanishes as the boundary of a 1-dimensional manifold must carry opposite orientation, we obtain $\iota^r \circ \partial - \partial \circ \iota^R = 0$. □

Let us now investigate the maps at the boundary ι^0 and ι^∞. For $R = \infty$, we have

$$\mathcal{A}_r^\infty(y',y) = \bigsqcup_{x\in\mathcal{O}(H^-)} \mathcal{B}(y',x) \times \mathcal{B}(x,y).$$

Remark 6.3.16. This space brings all the previously defined spaces together and is the reason for the, at the beginning very peculiar seeming, definitions of $\mathcal{B}(y',x)$ in Equation (5.2.7) and $\mathcal{B}(x,y)$ of Equation (6.2.10).

Lemma 6.3.17. *If $\mathrm{ind}(y) = \mathrm{ind}(y')$, we have that if both $\mathcal{B}(x,y)$ and $\mathcal{B}(y',x)$ are non-empty, then $\deg(x) = n - \mathrm{ind}(y)$.*

Proof. For $\mathrm{ind}(y) = \mathrm{ind}(y')$, we have that the dimension of $\mathcal{A}_r^\infty(y',y)$ is 0 and thus each $\mathcal{B}(x,y)$ and $\mathcal{B}(y',x)$ must also have dimension 0. Given that they are assumed to be non-empty, we have by Lemma 6.2.6 that

$$\dim(\mathcal{B}(x,y)) = \mathrm{ind}(y) + \deg(x) - n \overset{!}{=} 0.$$

Since $\dim(\mathcal{B}(y',x)) = -\mathrm{ind}(y') - \deg(x) + n$, we conclude. □

The moduli space $\mathcal{A}_r^\infty(y', y)$ therefore consists of the product of the moduli spaces used to define $\mathcal{V}_{r'}$ and $\mathcal{F}^r$. The choices made in [Abo14, Sec. 5.5.5.] to orient $\mathcal{A}_r^\infty(y', y)$ use the same orientation made in defining $\mathcal{V}_{r'}$ and $\mathcal{F}$, and so

$$\iota^\infty = \mathcal{V}_{r'} \circ \mathcal{F}^r \,. \tag{6.3.25}$$

For $R = 0$, Lemma 6.3.11 shows that the composition of $r' - r$ iterates of ι is homotopic to ι^0, up to an overall sign that is provided by Lemma 6.3.12. We split the Morse complex according to orientability of the piecewise geodesics induced by points of $\mathcal{L}^r M$:

$$CM_*(f^r; \eta) = CM_*^0(f^r; \eta) \otimes CM_*^{-1}(f^r; \eta).$$

Lemma 6.3.18 ([Abo14, Lem. 5.5.31.]). *The restriction of ι^0 to $CM_*^w(f^r; \eta)$ is homotopic to*

$$(-1)^{\frac{(n+w)(n+w-1)}{2}} \underbrace{\iota \circ \ldots \iota}_{r'-r}.$$

6.3.g) The Final Homotopy

All that is left is to provide a homotopy between ι^0 and ι^∞. Repeat the same procedure of the previous section with the parametrised moduli space $\mathcal{A}_r(y', y)$. Start with the isomorphism

$$|\det(\mathcal{A}_r(y', y))| \otimes o_y \cong o_{y'},$$

and assume that $\mathrm{ind}(y') = \mathrm{ind}(y) + 1$, making $\mathcal{A}_r(y', y)$ a 0-dimensional manifold. We obtain a map

$$\mathcal{H}_{(\gamma', w, \gamma)} : o_y \otimes \eta_y \to o_{y'} \otimes \eta_{y'}$$

for each element $(\gamma', w, \gamma) \in \mathcal{A}_r(y', y)$ as before. Define

$$\begin{aligned} \mathcal{H}^{\mathcal{V}\circ\mathcal{F}} &: CM_*(f^r; \eta) \to CM_*(f^{r'}; \eta) \\ \mathcal{H}^{\mathcal{V}\circ\mathcal{F}}\big|_{o_y \otimes \eta_y} &:= \bigoplus_{\mathrm{ind}(y') = \mathrm{ind}(y)+1} \sum_{(\gamma', w, \gamma) \in \mathcal{A}_r(y', y)} \mathcal{H}_{(\gamma', w, \gamma)}. \end{aligned} \tag{6.3.26}$$

Since $\mathcal{A}_r(\mathcal{L}^r M)$ provides a cobordism between $\mathcal{A}_r^0(\mathcal{L}^r M)$ and $\mathcal{A}_r^\infty(\mathcal{L}^r M)$, we have that $\mathcal{H}^{\mathcal{V}\circ\mathcal{F}}$ provides a homotopy between

$$\begin{aligned} \iota^0 &= \bigoplus_{\mathrm{ind}(y') = \mathrm{ind}(y)} \sum_{(\gamma', w, \gamma) \in \mathcal{A}_r^0(y', y)} \mathcal{H}_{(\gamma', w, \gamma)}, \quad \text{, and} \\ \iota^\infty &= \bigoplus_{\mathrm{ind}(y') = \mathrm{ind}(y)} \sum_{(\gamma', w, \gamma) \in \mathcal{A}_r^\infty(y', y)} \mathcal{H}_{(\gamma', w, \gamma)}, \end{aligned}$$

since they are the maps induced at the boundary of $\mathcal{A}_r(y', y)$. All the pieces needed to prove Proposition 6.3.1 are now in place, and proving it is a mere formality.

Proof of Prop. 6.3.1. Lemma 6.3.18 implies that ι^0 agrees with the $(r' - r)$ iterate of the inclusion map. Meanwhile, Equation (6.3.25) shows that ι^∞ is equal to $\mathcal{V}_{r'} \circ \mathcal{F}^r$. By the above discussion, the map $\mathcal{H}^{\mathcal{V}\circ\mathcal{F}}$ provides a homotopy between ι^0 and ι^∞. Taking this into consideration, together with the fact that ι^R is a chain map shows that ι^0 and ι^∞ agree on homology. □

6.4 The Isomorphism

In [Abo14, Sec. 6], Abouzaid shows the following theorem:

THEOREM 6.4.1 ([Abo14, Thm. 6.1.1.]). *$\mathcal{F}$ is surjective.*

This theorem then allows one to conclude the proof of Viterbo's theorem:

COROLLARY 6.4.2 ([Abo14, Cor. 6.1.2.]). *$\mathcal{F}$ and $\mathcal{V}$ are isomorphisms.*

Proof. $\mathcal{F}$ is an isomorphism because it is surjective and injective, see Corollary 6.0.2. We thus have by Theorem 6.4.1 that $\mathcal{V}$ is a left inverse to an isomorphism, implying that $\mathcal{V}$ is an isomorphism. □

The proof of Theorem 6.4.1 proceeds in a very similar manner to the one of the proof of Theorem 6.0.1. One starts by constructing maps

$$\mathcal{G}^r : HF^*(H;\mathbb{Z}) \to HM_{-*}(f^r;\eta)$$

from Floer cohomology to the Morse cohomology of finite dimensional approximations, using moduli spaces of discs with Lagrangian boundary conditions on cotangent fibres. Showing that these moduli spaces are compact requires a careful application of the integrated maximum principle, see [Abo14, Lem. 6.2.6].

As the proof is so similar, we will only give a broad overview[10] of the constructions, and refer to [Abo14, Sec. 6] for the details. The biggest hurdle that one needs to overcome in the proof of Theorem 6.4.1 is showing that there exists a cobordism between the moduli spaces defining $\mathcal{F}^r \circ \mathcal{G}^r$ and those defining the continuations map ([Abo14, Prop. 6.3.1.]). Since symplectic cohomology is defined as the filtered colimit over continuation maps, it follows from this cobordism that $\mathcal{F}$ is surjective.

The composition $\mathcal{F}^r \circ \mathcal{G}^r$ corresponds to pairs of discs with interior punctures with Lagrangian boundary conditions on cotangent fibres. In [Abo14, Sec. 6.3.2], Abouzaid glues such discs along the boundary to obtain the complement of r discs in the cylinder. There are then two crucial ideas that go into the proof of [Abo14, Prop. 6.3.1]: the first one is to construct a cobordism associated to degenerating these Riemannian surfaces to cylinders which are attached to r discs at interior points. The second insight is that the moduli space of holomorphic discs with boundary conditions on a given cotangent fibre consists only of constant discs. One can then take the union over all such fibres in order to represent T^*M as a parametrised moduli space of holomorphic discs. Attaching elements of this moduli space to a cylinder imposes no constraints, and thus we obtain the desired cobordism.

[10] Inspired by the overview of [Abo14, Sec. 6.1].

Chapter II

Viterbo's Theorem — the A_∞ Version

In the first chapter, we were able to establish a link between the symplectic cohomology of the cotangent bundle of a compact manifold M and the homology of free loop space $\mathcal{L}M$. Even though this link presents a profound result connecting Floer type invariants to classical invariants of the base, Viterbo's theorem inherits one major downside from its construction — it is defined at the level of (co-)homology. The study of homological mirror symmetry revealed that these (co-)homological considerations were too coarse, and that a purely chain-level result was needed.

First presented by Maxim Kontsevich at the ICM in Zürich [Kon98], homological mirror symmetry tries to explain mirror symmetry of a pair of Calabi-Yau manifold X and Y as predicted by string theory through an equivalence of derived categories. The conjecture states that the derived category of coherent sheaves on X is equivalent to the derived Fukaya category. If correct, homological mirror symmetry would thus reveal a deep connection between algebraic geometry and symplectic geometry. There exist many proven cases of this conjecture, the first non-trivial example was given by Polishchuk and Zaslow in [PZ00] for complex elliptic curves. Subsequently, Abouzaid and Smith showed its correctness in [AS09] for the 4-torus and in [Sei13], Seidel proved it for quartic surfaces. The preceding list is by no means exhaustive[1] and the conjecture remains open in the general case. We also refer to [Imp21] for a survey of the mathematical and physical origin of homological mirror symmetry.

A conjecture attributed to Arnold [Arn86] states that compact exact Lagrangian submanifolds inside the cotangent bundle should be Hamiltonian isotopic to the zero-section. In an attempt to prove this conjecture, ideas from homological mirror symmetry were used in [FSS07], and it became apparent that an understanding of Lagrangian Floer homology of the cotangent bundle at the chain level was needed, as one wants to immediately deal with the exact Lagrangians. In addition, in [Abo11] Abouzaid was able to show that a fibre generates the wrapped Fukaya category of a cotangent bundle, making fibres the most important objects to study.

[1] We refer to https://ncatlab.org/nlab/show/mirror+symmetry for an up-to-date list of completed proofs.

The second chapter will culminate in the proof of the following statement:

Theorem 2 ([Abo09, Thm. 1.1.]). *If M is a closed smooth manifold, there exists an A_∞-equivalence*

$$CW_b^*(T^*M) \to C_{-*}(\Omega_q M) \tag{6.4.1}$$

between the homology of the space of based loops $\Omega_q M$, and the Floer cohomology of the cotangent fibre at q with background class $b \in H^*(T^*M, \mathbb{Z}_2)$ given by the pullback of $w_2(M) \in H^*(M, \mathbb{Z}_2)$.

This theorem can be seen as the result of decades worth of investigating the deep relation between the symplectic topology of T^*M and the homology of the loop space of M.

7 Preliminaries for the Fukaya Category

The Fukaya category is a central object in modern symplectic geometry, and there exist many variants depending on the level of generality that one desires. As stated in [Aur13], all of these variations have common features: the objects are suitable Lagrangian submanifolds equipped with extra data, and the morphism spaces are given by Floer complexes endowed with the Floer differential. Composition of morphisms is given by the Floer product, which is however only associative up to homotopy. In fact, the Fukaya category is an incarnation of an algebraic structure called *A_∞-category*; implying that the differential and the product can be extended to a whole family of maps

$$\mu_k : \mathrm{Hom}(L_{k-1}, L_k) \otimes \cdots \otimes \mathrm{Hom}(L_0, L_1) \to \mathrm{Hom}(L_0, L_k)$$

satisfying certain compatibility conditions called *A_∞-relations*. Roughly speaking, these higher order maps will correspond to counting boundary punctured holomorphic discs and thus the Fukaya category provides a particularly pleasant approach to translate relations between such discs into the language of homological algebra. Even though the Fukaya category is such a ubiquitous object in mathematics, actually defining it requires us to lay some serious algebraic ground work. We will however make sure to justify the importance of each new concept introduced.

7.1 A_∞-Structures

A_∞-structures have their origin with J. Stasheff, who invented them at the beginning of the 1960's in order to investigate what higher coherence homotopies are needed to ensure that an H-space has a classifying space. In his 1994 ICM talk, M. Kontsevich conjectured that mirror symmetry could be explained as an equivalence of two triangulated categories that can both be expressed in terms of A_∞-structures. As stated before, this conjecture has since been proved in various settings, and thus cements A_∞-structures in contemporary mathematics. For the following introductory section we follow [Kel99] closely. For the later parts on A_∞-categories, we will adhere to the presentation given in [Sei08] and state almost all of the results without proof, as they would take us too far afield.

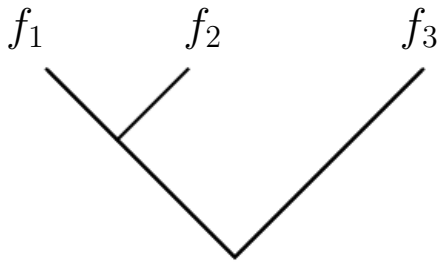
7.1.a) Origin of A_∞-Structures

Let (X, x) be a pointed topological space, and let $\Omega_x X$ be the based loop space. It consists of continuous maps $f : S^1 \to X$ such that $f(0) = x$. We have a continuous map

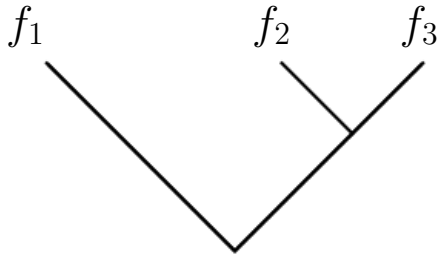
$$\begin{aligned} m_2 : \Omega_x X \times \Omega_x X &\to \Omega_x X \\ (f_1, f_2) &\mapsto f_1 * f_2. \end{aligned}$$

Strictly speaking, this composition is not associative: given another loop f_3, the composition $(f_1 * f_2) * f_3$ does not have to equal $f_1 * (f_2 * f_3)$ as for example $((f_1 * f_2) * f_3)(\frac{1}{4}) = f_1(1)$, but $(f_1 * (f_2 * f_3))(\frac{1}{4}) = f_1(\frac{1}{2})$. These different compositions can be encoded in a combinatorial manner using binary trees with leaves f_1, f_2 and f_3.

$(f_1 * f_2) * f_3 \longrightarrow$



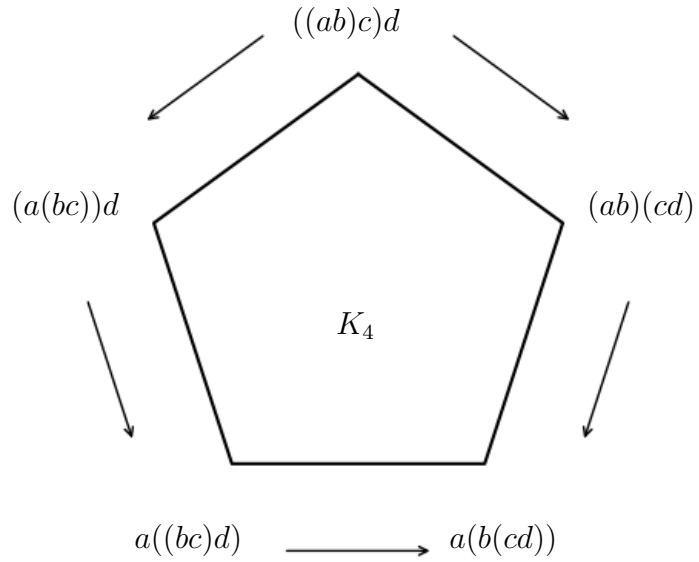
$f_1 * (f_2 * f_3) \longrightarrow$



A combinatorially equivalent way to encode this are the different parenthesisations of the word *abc*. Even though different combinations of m_2 are not associative, there clearly exists a homotopy

$$m_3 : [0, 1] \times \Omega_x X \times \Omega_x X \to \Omega_x X$$

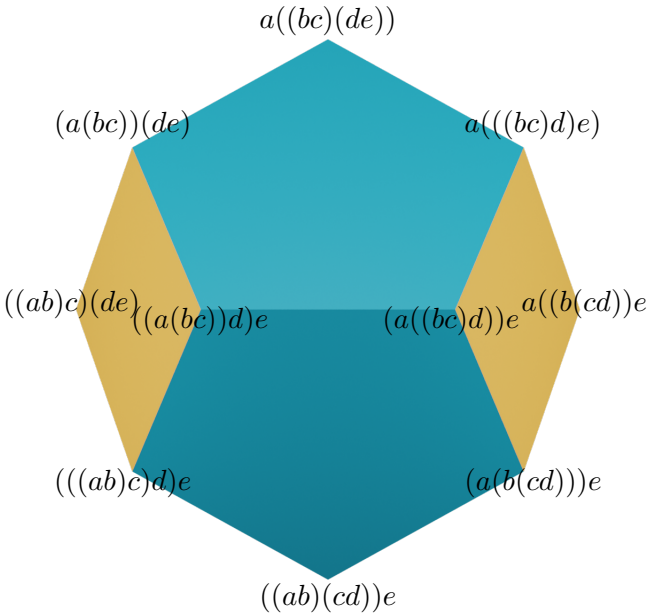
between the two possibilities. When there are 4 factors, there are 5 combinatorial variations of combinations of m_2, each one corresponding to one of the 5 parenthesisations of the word *abcd*.

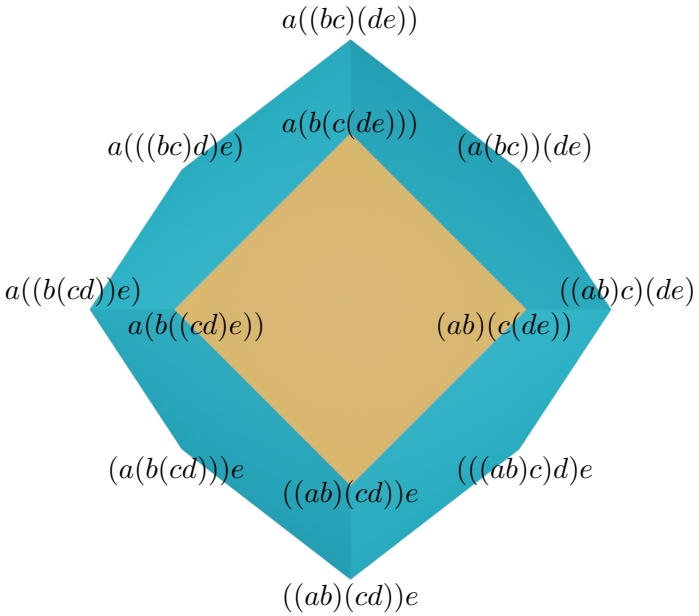


Using m_3, we obtain two paths of homotopies joining the composition $((f_1 * f_2) * f_3) * f_4$ to $f_1 * (f_2 * (f_3 * f_4))$, i.e. by traversing the pentagon K_4 clock- respectively counter-wise. These two paths then again themselves are homotopic:

$$m_4 : K_4 \times (\Omega_x X)^4 \to \Omega_x X.$$

Composing 5 factors yields 14 possibilities. Using m_3 and m_4, we obtain paths linking the compositions and faces linking the paths. The resulting geometrical object encoding these relations is the Stasheff polytope K_5.





Where the pentagonal faces are copies of K_4, the figure on the left is a front view of K_5 and the right figure a view of the back. However, this is where our ability to visualise compositions stops, as in general K_n is a polytope of dimension $n-2$

for $n \geq 2$. Stasheff then defined an A_∞- space to be a topological space Y together with maps

$$m_n : K_n \times Y^n \to Y$$

that satisfy certain compatibility conditions and admit something called a *strict unit*. To no surprise, the base looped space $\Omega_x X$ is then an A_∞-space.

7.1.b) A_∞-Algebras

As it turns out, the singular chain complexes of A_∞-spaces present interesting algebraic properties.

DEFINITION 7.1.1. *Let K be a field. An A_∞-**algebra** over K is a $\mathbb{Z}$-graded vector space*

$$A = \bigoplus_{p \in \mathbb{Z}} A^p$$

endowed with graded maps

$$m_n : A^{\otimes n} \to A \ \ , n \geq 1$$

of degree $2 - n$ satisfying the following relations:

i) $m_1 \circ m_1 = 0$, *i.e. (A, m_1) is a differential complex,*

ii) $m_1 \circ m_2 = m_2(m_1 \otimes \mathrm{id} + \mathrm{id} \otimes m_1)$, *i.e. m_1 is a graded derivative with respect to the multiplication m_2.*

iii) $m_2 \circ (\mathrm{id} \otimes m_2 - m_2 \otimes \mathrm{id}) = m_1 m_3 + m_3(m_1 \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes m_1 \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \otimes m_1)$.

iv) In general, for $n \geq 1$, we have

$$\sum_{n=r+s+t} (-1)^{r+st} m_u(\mathrm{id}^{\otimes r} \otimes m_2 \otimes \mathrm{id}^{\otimes t}) = 0,$$

where $u = r + 1 + t$.

We have two remarks concerning this definition.

REMARK 7.1.2. When we apply these compatibility conditions to elements, additional signs appear by the Koszul sign rule:

$$(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y),$$

where f and g are graded maps and x and y are homogeneous elements.

REMARK 7.1.3. One can make sense of these compatibility relations by recalling that given two chain complexes in the category of graded vector spaces $(A_\bullet, d_A)$ and $(B_\bullet, d_B)$, there exists a graded internal Hom of chain complexes defined by setting

$$\mathrm{Hom}_i^n(A, B) := \{\text{homogeneous } f : A_i \to B_{i+n} \mid f \text{ is of degree } n\}$$

together with a differential

$$d(f) = d_B \circ f - (-1)^{|f|} f \circ d_A.$$

Then we can identify Relation ii) with

$$\underbrace{m_1}_{d_A} \circ m_2 = m_2 \circ \underbrace{(m_1 \otimes \mathrm{id} + \mathrm{id} \otimes m_1)}_{d_{A\otimes A}}$$

and iii) with

$$\underbrace{m_2 \circ (\mathrm{id} \otimes m_2 - m_2 \otimes \mathrm{id})}_{\substack{a*(b*c)-(a*b)*c \\ \text{associator for } m_2 = *}} = \underbrace{m_1 m_3 + m_3(m_1 \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes m_1 \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \otimes m_1)}_{d(m_3) \text{ in } \mathrm{Hom}_k^\bullet(A^{\otimes 3}, A)}.$$

Similarly, commutators of higher operations correspond to boundaries of higher operations in the internal Hom. This makes precise the notion of 'associativity up to homotopy'.

7.1.c) A_∞-Categories

The above ideas readily generalise to a more categorical setting.

Definition 7.1.4. *Fix a coefficients field $\mathcal{K}$. All of the categories will be assumed to be linear over $\mathcal{K}$ and small. An A_∞-**category** $\mathcal{A}$ consists of a set of objects $\mathrm{Obj}(\mathcal{A})$, together with a graded vector space $\mathrm{Hom}_{\mathcal{A}}(X_0, X_1)$ for any pair of objects, and composition maps*

$$\mu^d_{\mathcal{A}} : \mathrm{Hom}_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{A}}(X_0, X_1) \to \mathrm{Hom}_{\mathcal{A}}(X_0, X_d)$$

of degree $d-2$. In addition, the maps must satisfy the A_∞-relations

$$\sum_{\substack{1\le m\le d \\ 0\le n\le d-m}} (-1)^{\spadesuit_n} \mu^{d-m+1}_{\mathcal{A}}(a_d, \ldots, a_{n+m+1}, \mu^m_{\mathcal{A}}(a_{n+m}, \ldots, a_{n+1}), a_n, \ldots, a_1) = 0, \tag{7.1.1}$$

where $\spadesuit_n := |a_1| + \cdots + |a_n| - n$.

Remark 7.1.5. Note that we chose to immediately include the Koszul signs this time. In general, an A_∞-category might not be a category. The reason is twofold: the composition $\mu^2_{\mathcal{A}}$ is not necessarily associative; additionally, the existence of identity morphisms is not given by the definition. One pesky consequence of this fact is that there is no intrinsic notion of isomorphic objects. However, full and faithful functors still make sense.

Remark 7.1.6. If $\mathcal{A}$ only has one object $*$, then it is equivalent to the endomorphism A_∞-algebra $\mathrm{Hom}_{\mathcal{A}}(*,*)$. Furthermore, if $\mu^d_{Ac} = 0$ for all $d > 2$, then $\mathcal{A}$ is a dg-category, and vice versa.

Definition 7.1.7. *Let $\mathcal{A}$ be an A_∞-category and $A \in \mathrm{Obj}(\mathcal{A})$. A morphism $e_A \in \mathrm{Hom}_{\mathcal{A}}{}^0(A, A)$ is called a **strict identity** if we have*

$$m_2(f, e_A) = f \quad \text{and} \quad m_2(e_A, g) = g$$

whenever these make sense, and

$$m_n(\ldots, e_A, \ldots) = 0 \quad \text{for all } n \neq 2.$$

*If e_A exists, it is unique. We say that an A_∞-category has **strict identities** if there is a strict identity e_A for each object $A \in \mathrm{Obj}(\mathcal{A})$.*

To every A_∞-category $\mathcal{A}$, we can associate the cohomological category $H(\mathcal{A})$: its objects are the same as the ones of $\mathcal{A}$, morphism spaces are the cohomological groups $H(\mathrm{Hom}_{\mathcal{A}}(X_0, X_1), \mu^1_{\mathcal{A}})$ and composition is given by

$$[a_2] \cdot [a_1] = (-1)^{|a_1|}[\mu^2_{\mathcal{A}}(a_2, a_1)].$$

Except for a possible lack of identity morphism, we can in this way recover an ordinary linear graded category. Note that if e_A is a strict unit for an object $A \in \mathrm{Obj}(\mathcal{A})$, we have that e_A is a cycle of the complex $(\mathrm{Hom}_{\mathcal{A}}(A, A), m_1)$. Even though the following definition is a bit technical, having a notion of functors between A_∞-structures will be useful later on.

Definition 7.1.8. *An A_∞-**functor** between two A_∞-categories $\mathcal{A}$ and $\mathcal{B}$ consists of a map $\mathcal{F} : \mathrm{Obj}(\mathcal{A}) \to \mathrm{Obj}(\mathcal{B})$ and multilinear maps of all orders $d \geq 1$,*

$$\mathcal{F}^d : \mathrm{Hom}_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{A}}(X_0, X_1) \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}(X_0), \mathcal{F}(X_d))$$

of degree $1 - d$, and for $X_0, \ldots, X_d \in \mathrm{Obj}(\mathcal{A})$. These maps shall in addition satisfy the following equations:

$$\begin{aligned}
&\sum_{r \geq 1} \sum_{s_1 + \cdots + s_r = d} \mu^r_{\mathcal{B}}(\mathcal{F}^{s_r}(a_d, \ldots, a_{d-s_r+1}), \ldots, \mathcal{F}^{s_1}(a_{s_1}, \ldots, a_1)) \\
&= \sum_{\substack{1 \leq m \leq d \\ 0 \leq n \leq d-m}} (-1)^{\spadesuit_n} \mathcal{F}^{d-m+1}(a_d, \ldots, a_{n+m+1}, \mu^m_{\mathcal{A}}(a_{n+m}, \ldots, a_{n+1}), a_n, \ldots, a_1).
\end{aligned}$$

If $d = 1$, then the above equation boils down to $\mathcal{F}^1$ being a chain map, i.e. that $\mu^1_{\mathcal{B}}(\mathcal{F}^1(a_1)) = \mathcal{F}^1(\mu^1_{\mathcal{A}}(a_1))$. If $d = 2$, then the above equation becomes:

$$\begin{aligned}
&\mu^2_{\mathcal{B}}(\mathcal{F}^1(a_2), \mathcal{F}^1(a_1)) - \mathcal{F}^1(\mu^2_{\mathcal{A}}(a_2, a_1)) \\
&= \mu^1_{\mathcal{B}}(\mathcal{F}^2(a_2, a_1)) - \mathcal{F}^2(\mu^1_{\mathcal{A}}(a_2), a_1) - \mathcal{F}^2(a_2, \mu^1_{\mathcal{A}}(a_1)).
\end{aligned}$$

Roughly speaking, the left side quantifies the failure of $\mathcal{F}^1$ to commute with the compositions, and the right side represents the failure of μ^1 to be a derivation (up to signs). This shows that A_∞-structures really are able to intertwine the different degrees of operations defined on them.

On a different note, there is an induced functor $H(F) : H(\mathcal{A}) \to H(\mathcal{B})$, whose action on morphisms is $[a] \mapsto [\mathcal{F}^1(a)]$. We say that $\mathcal{F}$ is a *quasi-isomorphism* if it is an isomorphism of categories. We say that $\mathcal{F}$ is *cohomologically full and faithful* if $H(\mathcal{F})$ is full and faithful.

A_∞-functors $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ are themselves the objects of an A_∞- category $\mathcal{F}un(\mathcal{A}, \mathcal{B})$. An element $T \in \mathrm{Hom}^g_{\mathcal{F}un(\mathcal{A},\mathcal{B})}(\mathcal{F}_0, \mathcal{F}_1)$ of the chain space of morphisms in this category is a sequence $(T^0, T^1, \ldots)$, where each T^d is a family of multilinear maps

$$\mathrm{Hom}_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{A}}(X_0, X_1) \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}_0(X_0), \mathcal{F}_1(X_d))$$

of degree $g - d$; for all $(X_0, \ldots, X_d)$. These multilinear maps should in addition satisfy compatibility conditions similar to (7.1.1), for which we refer to [Sei08, Sec. (1d)].

7.1.d) Yoneda Embedding

The Yoneda Lemma is one of the most important advancements in category theory and it was K. Fukaya's insight in the late 90's that introduced it to symplectic geometry. This lemma enables one to embed any locally small category into a category of functors defined on that category. Heuristically, the Yoneda lemma is able to describe objects by taking into account all their relations to other objects. In order to give this functor space structure, we need to introduce the notion of A_∞-*modules*.

Definition 7.1.9. *Let $\mathcal{C}h$ be the differential graded category of chain complexes of $\mathbb{K}$-vectors, considered as an A_∞-category. For a fixed A_∞-category $\mathcal{A}$, define the A_∞-category of (right) A_∞-**modules** over $\mathcal{A}$ to be*

$$\mathcal{M}od(\mathcal{A}) = \mathcal{F}un(\mathcal{A}^{\mathrm{op}}, \mathcal{C}h).$$

In particular, this means that an object $\mathcal{M}$ of $\mathcal{M}od(\mathcal{A})$ consist of a graded vector space $\mathcal{M}(X)$ for every $X \in \mathrm{Obj}(\mathcal{A})$ and maps

$$\mu^d_{\mathcal{M}} : \mathcal{M}(X_{d-1}) \otimes \mathrm{Hom}_{\mathcal{A}}(X_{d-2}, X_{d-1}) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{A}}(X_0, X_1) \to \mathcal{M}(X_0)$$

of degree $2-d$, in accordance with the compatibility conditions for functors alluded to previously.

Definition 7.1.10. *Let $\mathcal{M}_0$ and $\mathcal{M}_1$ be two A_∞-modules. **Pre-module homomorphisms** are then morphisms in $\mathrm{Hom}_{\mathcal{M}od(\mathcal{A})}(\mathcal{M}_0, \mathcal{M}_1)$ that can be written as sequences*

$$t^d : \mathcal{M}_0(X_{d-1}) \otimes \mathrm{Hom}_{\mathcal{A}}(X_{d-2}, X_{d-1}) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{A}}(X_0, X_1) \to \mathcal{M}_1(X_0)$$

of maps of degree $|t| - d + 1$. The actual module homomorphisms are then those t that in addition satisfy $\mu^1_{\mathcal{M}od(\mathcal{A}_H)}(t) = 0$.

The explicit action of the differential and composition of such pre-module homomorphisms on elements is given in [Sei08, Eq. (1.21.)].

We are now in the position to formally state the Yoneda embedding. Given an object Y of an A_∞-category $\mathcal{A}$, we can assign an A_∞-module $\mathcal{Y}$ to it by setting $\mathcal{Y}(X) := \mathrm{Hom}_{\mathcal{A}}(X, Y)$ and $\mu^d_{\mathcal{Y}} = \mu^d_{\mathcal{A}}$. This construction is part of a canonical A_∞-functor

$$よ_{\mathcal{A}}{}^{2} : \mathcal{A} \to \mathcal{M}od(\mathcal{A}),$$

called the Yoneda embedding. For $c \in \mathrm{Hom}_{\mathcal{A}}(Y_0, Y_1)$, $よ^1(c) \in \mathrm{Hom}_{\mathcal{M}od(\mathcal{A})}(\mathcal{Y}_0, \mathcal{Y}_1)$ is then the pre-module homomorphism

$$\begin{aligned}\mathcal{Y}_0(X_{d-1}) \otimes \mathrm{Hom}_{\mathcal{A}}(X_{d-2}, X_{d-1}) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{A}}(X_0, X_1) &\to \mathcal{Y}_1(X_0),\\ (b, a_{d-1}, \ldots, a_1) &\mapsto \mu^{d+1}_{\mathcal{A}}(c, b, a_{d-1}, \ldots, a_1).\end{aligned}$$

[2] Where よ is the Japanese hiragana character for 'yo'.

For $d \geq 2$, the higher order terms $ょ^d$ are then defined analogously to this formula. For any A_∞-module $\mathcal{M}$, one can generalise $ょ^1$ to a chain map

$$\lambda_{\mathcal{M}} : \mathcal{M}(Y) \to \mathrm{Hom}_{\mathcal{M}od(\mathcal{A})}(\mathcal{Y}, \mathcal{M}),$$
$$\lambda(c)^d(b, a_{d-1}, \ldots, a_1) = \mu^{d+1}_{\mathcal{M}}(c, b, a_{d-1}, \ldots, a_1).$$

When an A_∞-module $\mathcal{M}$ presents a strict unit (as an A_∞-category), we are able to say a lot more:

LEMMA 7.1.11 ([Sei08, Lem. 2.12.]). *For any A_∞-module $\mathcal{M}$ with strict identities, and any object Y of $\mathcal{A}$, the map λ from above is a quasi-isomorphism.*

$\square_{\text{ref.}}$

Specialising to A_∞-modules $\mathcal{M}$ that come from the Yoneda embedding, we have $\lambda = ょ^1$, and thus:

COROLLARY 7.1.12 ([Sei08, Cor. 2.13.]). *$ょ$ is cohomologically full and faithful.*

Because of the importance of this Lemma 7.1.11, we will from now on assume that all A_∞-categories present strict identities. In the next section, we will utilise this heavy algebraic machinery to generalise the mapping cone constructions from homological algebra to A_∞-categories.

7.2 Exact Triangles

Briefly taking a step back, we realise that the existence of an A_∞-structure actually claims a lot of data; namely a set of objects together with an infinite amount of higher order associativity relations between them. This makes A_∞-categories hard to control in theoretical considerations, and certainly also in explicit computations. Luckily, there is a homological algebraic tool that can help us get a grip on A_∞-structures — exact triangles. With these objects, it is sometimes possible to understand the whole A_∞-category solely by considering a small, preferably finite, set of generators.

7.2.a) Mapping Cones

Mapping cones have their origin in the realm of topology where they have found many applications over the years. Given a map $f : X \to Y$, the mapping cone $C(f)$ is defined to be the quotient space of the mapping cylinder $(X \times Y) \sqcup_f Y$ under the equivalence relation

$$(X, 0) \sim (X', 0) \quad \text{and} \quad (X, 1) \sim f(x).$$

There then exists a sequence of maps

$$X \xrightarrow{f} Y \xrightarrow{i} C(f) \xrightarrow{p} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \longrightarrow \ldots \tag{7.2.1}$$

where i is the inclusion of Y into the mapping cone, and p is the projection onto ΣX obtained by collapsing Y, this works as all the points of $X \times \{1\}$ are then identified with a point. The suspension functor Σ can be seen as a first incarnation

of something that will later be called a *shift functor*.

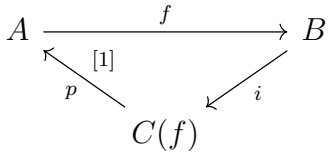
In homological algebra, there exists a mapping cone that is directly modelled on the topological one. Considering chain complexes over an abelian category, so that taking about (co-)homology makes sense, we have that if the cone of a map $f : A \to B$ between two chain complexes (A, d_A) and (B, d_B) is acyclic (with respect to a to be defined differential), then f is actually a quasi-isomorphism. Define the chain complex

$$\begin{aligned} C^n(f) &:= A^{n+1} \oplus B^n \\ d_{C(f)} &:= \begin{pmatrix} d_{A[1]} & 0 \\ f[1] & d_B \end{pmatrix}, \end{aligned} \tag{7.2.2}$$

where $A[1]$ denotes the shifted complex, i.e. $A[1]^n = A^{n+1}$ and $d^n_{A[1]} = -d^{n+1}_A$. The mapping cone then gets its name from the fact that it fits into a similar sequence to (7.2.1)

$$A \xrightarrow{f} B \xrightarrow{i} C(f) \xrightarrow{p} A[1] \tag{7.2.3}$$

where i is given by the inclusion and p by the projection into/from the direct sum $C(f)$. The reason why this is called a triangle is that it neatly fits into the diagram



where $p : C(f) \to A[1]$. As it turns out, this triangle induces a long exact sequence in homology:

$$\ldots \longrightarrow H_{i+1}(C(f)) \longrightarrow H_i(A) \xrightarrow{H_i(f)} H_i(B) \longrightarrow H_i(C(f)) \longrightarrow \ldots$$

if $C(f)$ is acyclic, this shows that $H_i(A) \cong H_i(B)$ and thus that f is a quasi-isomorphism.

7.2.b) Exact Triangles in A_∞-Categories

The homological mapping cone now serves as inspiration to translate exact triangles to A_∞-categories.

Definition 7.2.1. *Let Y_0 and Y_1 be objects of an A_∞-category $\mathcal{A}$ and $c \in \mathrm{Hom}_{\mathcal{A}}{}^0(Y_0, Y_1)$ be a cocycle, i.e. $\mu^1_{\mathcal{A}}(c) = 0$. The **abstract mapping cone** of c is the A_∞-module $\mathcal{C}one(c)$ defined by*

$$\mathcal{C}one(c)(X) := \mathrm{Hom}_{\mathcal{A}}(X, Y_0)[1] \oplus \mathrm{Hom}_{\mathcal{A}}(X, Y_1), \tag{7.2.4}$$

$$\begin{aligned} &\mu^d_{\mathcal{C}one(c)}((b_0, b_1), a_{d-1}, \ldots, a_1) \\ &\qquad := \Big(\mu^d_{\mathcal{A}}(b_0, a_{d-1}, \ldots, a_1), \mu^d_{\mathcal{A}}(b_1, a_{d-1}, \ldots, a_1) + \mu^{d+1}_{\mathcal{A}}(c, b_0, a_{d-1}, \ldots, a_1)\Big). \end{aligned}$$

The fact that this is actually an A_∞-module is shown in [Sei08, Sec.(3e)]. These abstract mapping cones with canonical module homomorphisms $\iota \in \mathrm{Hom}^0_{\mathcal{M}od(\mathcal{A})}(\mathcal{Y}_1, \mathcal{C}one(c))$ and $\pi \in \mathrm{Hom}^1_{\mathcal{M}od(\mathcal{A})}(\mathcal{C}one(c), \mathcal{Y}_0)$, which one should think of as inclusion and projection respectively. These two maps are given in terms of (7.2.4) by

$$\begin{aligned} \iota^1(b_1) &= (0, (-1)^{|b_1|} b_1) \quad \text{, and} \\ \pi^1(b_0, b_1) &= (-1)^{|b_0|-1} b_0, \end{aligned}$$

with vanishing higher order terms. These form a triangle in the homology category $H(\mathcal{M}od(\mathcal{A}))$:

$$\begin{array}{ccc} \mathcal{Y}_0 & \xrightarrow{[l^1(c)]} & \mathcal{Y}_1 \\ & {\scriptstyle [\pi]} \nwarrow \quad \swarrow {\scriptstyle [\iota]} \; {\scriptstyle [1]} & \\ & \mathcal{C}one(c) & \end{array} \tag{7.2.5}$$

where [1] indicates that the map $[\pi]$ has associated degree 1. The Yoneda embedding now provides the key to the link to A_∞-categories. Recall that for an object Y of an A_∞-category $\mathcal{A}$ it was defined by setting $\mathcal{Y}(X) = \mathrm{Hom}_{\mathcal{A}}(X, Y)$ and $\mu^d_{\mathcal{Y}} = \mu^d_{\mathcal{A}}$. We say that an A_∞-module $\mathcal{M}$ is *quasi-represented* by Y if there exists an isomorphism between $\mathcal{Y}$ and $\mathcal{M}$ in $H(\mathcal{M}od(\mathcal{A}))$.

Definition 7.2.2. *Let $\mathcal{Y}_0$ and $\mathcal{Y}_1$ be objects of $\mathcal{M}od(\mathcal{A})$ and $c \in \mathrm{Hom}^0_{\mathcal{M}od(\mathcal{A})}(\mathcal{Y}_0, \mathcal{Y}_1)$ a degree zero cocyle. Let $\mathcal{C}one(c)$ be the abstract mapping cone of Definition 7.2.1, and let Y_0, Y_1 and Y_2 be objects of $H(\mathcal{A})$. We say that Y_0, Y_1 and Y_2 form an **exact triangle** if they quasi-represent $\mathcal{Y}_0, \mathcal{Y}_1$ and $\mathcal{C}one(c)$ respectively, and there exists a triangle*

$$\begin{array}{ccc} Y_0 & \xrightarrow{[c_1]} & Y_1 \\ & {\scriptstyle [c_3]} \nwarrow \quad \swarrow {\scriptstyle [c_2]} & \\ & Y_2 & \end{array}$$

which becomes isomorphic to (7.2.5) under the Yoneda embedding.

There exist various more concrete characterisations for when a triangle is exact, but as we will never explicitly use them we refer for example to [Sei08, Lem. 3.7.] and [Sei08, Prop. 3.8.].

7.3 Triangulated Categories

Morally speaking, triangulated categories are categories that possess cones and a notion of a shift functor[3]. Given an object $Y \in \mathrm{Obj}(\mathcal{A})$, we can for $\sigma \in \mathbb{N}$ define the *σ-fold shift* of Y by setting $S^\sigma Y := \mathbb{K}[\sigma] \otimes Y$, where $\mathbb{K}[\sigma]$ is the 1-dimensional vector space supported in degree $-\sigma$, and the tensor product is defined in the obvious way [Sei08, Sec.(3c)].

Definition 7.3.1. *We say that an A_∞-category $\mathcal{A}$ is **triangulated** if:*

1. *it is non-empty,* $\mathrm{Obj}(\mathcal{A}) \neq \varnothing$,

[3] As we alluded to previously.

2. *every morphism* $[c_1]$ *in* $H^0(\mathcal{A})$ *can be completed to an exact triangle,*

3. *for any object* Y*, there exists another object* $\widetilde{Y}$ *such that* $S\widetilde{Y} \cong Y$ *in* $H^0(\mathcal{A})$*.*

A reader familiar with ordinary triangulated categories might be wondering why we are not imposing addition axioms on the exact triangles, as first given in their total form in [Ver67, Ch. 2]. This is due to the fact that the A_∞-structure is actually rich enough to "know" about these axioms and analogues of them can be derived, see [Sei08, Cor. 3.15-3.19]. Moreover, we have the following two remarkable results:

PROPOSITION 7.3.2 ([Sei08, Cor. 3.9.]). *If* $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ *is an* A_∞*-functor, then* $H(\mathcal{F})$ *maps exact triangles in* $H(\mathcal{A})$ *to exact triangles in* $H(\mathcal{B})$*.*

$\square_{\text{ref.}}$

PROPOSITION 7.3.3 ([Sei08, Prop. 3.14.]). *If* $\mathcal{A}$ *is a triangulated* A_∞*-category, then* $H^0(\mathcal{A})$ *is an ordinary triangulated category.*

$\square_{\text{ref.}}$

EXAMPLE 7.3.4. As it turns out, we have already constructed one such triangulated A_∞-category—the A_∞-category $\mathcal{M}od(\mathcal{A})$. This follows immediately from the fact that we were able to give an explicit description of abstract mapping cones.

Next, we specify what we meant by saying that exact triangles can help us identify subsets that generate the entire category. Let $\mathcal{B}$ be a triangulated A_∞-category, and $\mathcal{A} \subset \mathcal{B}$ a full A_∞-subcategory, which is non-empty. Take the smallest full subcategory $\widetilde{\mathcal{B}} \subset \mathcal{B}$ that is characterised as follows: $\widetilde{\mathcal{B}}$ is assumed to be closed under isomorphisms, i.e. if $X_0 \in \mathrm{Obj}(\widetilde{B})$, then if $X_1 \in \mathrm{Obj}(B)$ is isomorhpic to X_0 in $H^0(\mathcal{B})$, we have that $X_1 \in \mathrm{Obj}(\widetilde{\mathcal{B}})$; furthermore $\widetilde{\mathcal{B}}$ is assumed to be itself triangulated.

DEFINITION 7.3.5. *Let* $\mathcal{A} \subset \mathcal{B}$ *be as above. We say that the triangulated subcategory* $\widetilde{\mathcal{B}}$ *is* ***generated*** *by* $\mathcal{A}$*. We say that* $\mathcal{A}$ ***generates*** $\mathcal{B}$ *if* $\mathcal{B} = \widetilde{\mathcal{B}}$*.*

Note that $\widetilde{\mathcal{B}}$ can be explicitly constructed by taking all possible mapping cones and shifts, and then repeating this an arbitrary amount of times.

DEFINITION 7.3.6. *Let* $\mathcal{A}$ *be a non-empty* A_∞*-category. A* ***triangulated envelope*** *is a pair* $(\mathcal{B}, \mathcal{F})$ *consisting of i) a triangulated* A_∞*-category* $\mathcal{B}$ *and ii) a cohomologically full and faithful functor* $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ *such that the objects in the image of* $\mathcal{F}$ *are generators for* $\mathcal{B}$*.*

By using the fact that the Yoneda embedding is a full and faithful functor, together with the fact that $\mathcal{M}od(\mathcal{A})$ is triangulated, we obtain that triangulated envelopes always exist by taking the full subcategory of $\mathcal{M}od(\mathcal{A})$ generated by the image of the Yoneda embedding. Moreover, envelopes are to some extent equivalent:

PROPOSITION 7.3.7 ([Sei08, Lem. 3.20.]). *Triangulated envelopes always exist. Moreover, if* $(\mathcal{B}, \mathcal{F})$ *and* $(\widetilde{\mathcal{B}}, \widetilde{\mathcal{F}})$ *are two such envelopes for* $\mathcal{A}$*, then there is a quasi-equivalence* $\mathcal{G} : \mathcal{B} \to \widetilde{\mathcal{B}}$ *such that* $\mathcal{G} \circ \mathcal{F}$ *is isomorphic to* $\widetilde{\mathcal{F}}$ *in* $H^0(\mathcal{F}un(\mathcal{A}, \mathcal{B}))$*.*

□ref.

The upshot of this result is that if we are able to construct a particularly convenient envelope, then it will be indistinguishable from others for all intents and purposes. One exceptionally nice choice of envelope are *twisted complexes*, as they are able to retain a lot of the information about the original category. In addition, the mapping cone construction can be explicitly given, which makes twisted complexes some of the easiest envelopes to handle in applications.

Definition 7.3.8. *A* ***twisted complex*** (E, δ^E) *consists of:*

- *a formal direct sum* $E = \oplus_{i=1}^N E_i[k_i]$ *of shifted objects of* $\mathcal{A}$,
- *a strictly lower triangulated differential* $\delta^E \in \mathrm{End}^1(\mathrm{E})$, *meaning a collection of maps* $\delta_{ij}^E \in \mathrm{Hom}^{k_j - k_i + 1}(E_i, E_j)$, $1 \leq i < j \leq N$ *satisfying*

$$\sum_{k \geq 1} \mu^k(\delta^E, \dots, \delta^E) = 0 \quad \text{, or more explicitly}$$
$$\sum_{k \geq 1} \sum_{i = i_0 < i_1 < \cdots < i_k = j} \mu^k(\delta^E_{i_{k-1}, i_k}, \dots, \delta^E_{i_0, i_1}) = 0 \quad \text{for all } 1 \leq i < j \leq N.$$

A ***degree d morphism*** *of two twisted complexes* (E, δ^E) *and* $(E', \delta E')$ *is a degree* d *map between the direct sums. Explicitly, a morphism* $a \in \mathrm{Hom}^d(E, E')$ *is a collection of maps* $a_{ij} \in \mathrm{Hom}^{d + k'_j - k_i}(E_i, E'_j)$, *where we needed to adjust the degree by* $k'_j - k_i$ *to take the shifts into account.*

In order to make twisted complexes into an A_∞*-category, we set*

$$\mu^k_{\mathrm{Tw}}(a_k, \dots, a_1) = \sum_{j_0, \dots, j_k \geq 0} \mu^{k + j_0 + \cdots + j_k}(\underbrace{\delta^k, \dots, \delta^k}_{j_k}, a_k \dots, \underbrace{\delta^1, \dots, \delta^1}_{j_1}, a_1, \underbrace{\delta^0, \dots, \delta^0}_{j_0}), \tag{7.3.1}$$

for twisted complexes (E_i, δ^i) *and morphisms* $a_i \in \mathrm{Hom}(E_{i-1}, E_i)$.

Remark 7.3.9. Note that the sum (7.3.1) is finite since each δ^i is strictly lower triangular.

Proposition 7.3.10 ([Sei08, Lem. 3.28.] & [Sei08, Lem. 3.32.]). $\mathrm{Tw}(\mathcal{A})$ *is generated by its full subcategory* $\mathcal{A}$, *where we view each object in* $\mathcal{A}$ *as its own twisted complex in grading 0. Moreover,* $\mathrm{Tw}(\mathcal{A})$ *is a triangulated* A_∞*-category.*

□ref.

This makes $\mathrm{Tw}(\mathcal{A})$ a triangulated envelope for $\mathcal{A}$, and in addition we have the following interesting result:

Proposition 7.3.11 ([Sei08, Lem. 3.33.]). *An* A_∞*-category* $\mathcal{A}$ *is triangulated if and only if the embedding* $\mathcal{A} \hookrightarrow \mathrm{Tw}(\mathcal{A})$ *is a quasi-equivalence.*

□ref.

The following definition gives an explicit description of the mapping cone construction in analogy to the homological mapping cones (7.2.2).

Definition 7.3.12. *Given twisted complexes (E, δ) and (E', δ') and a degree zero cocylce $c \in \mathrm{Hom}^0_{\mathrm{Tw}(\mathcal{A})}(E, E')$, we define the* ***abstract mapping cone*** *of f to be the twisted complex*

$$\mathrm{Cone}(c) = \left(E[1] \oplus E', \begin{pmatrix} \delta & 0 \\ c & \delta' \end{pmatrix} \right).$$

In conclusion, we can assertively say that twisted complexes are our preferred triangulated envelopes for any given A_∞-category. The enlargements that go into the definition of twisted complexes naturally encode shifts (through the formal direct sum construction) and, most importantly, an explicit description of the mapping cones inspired by the mapping cone construction in homological algebra.

8 Definition of the A_∞-Categories

In this section, we introduce the A_∞-categories needed to make sense of Equation 6.4.1.

8.1 The Pontryagin Category

The following construction presents many similarities to the one in 7.1.a), but is different in a key point. For M a path-connected topological space, consider the topological category composed of objects the points in M, and morphisms between $q_0, q_1 \in M$ given by the Moore path space:

$$\Omega(q^0, q^1) := \{\gamma : [0, R] \to M \mid \gamma(0) = q^0 \text{ and } \gamma(1) = q^1\},$$

where crucially, R is allowed to lie in $[0, +\infty)$. This seemingly minor difference to the construction in Section 7.1.a) has the major consequence that the composition

$$\begin{aligned}\Omega(q^0, q^1) \times \Omega(q^1, q^2) &\to \Omega(q^0, q^2)\\ (\gamma_1, \gamma_2) &\mapsto \gamma_1 \cdot \gamma_2(t) = \begin{cases} \gamma_1(t) & \text{if } 0 \le t \le R_1 \\ \gamma_2(t - R_1) & \text{if } R_1 \le t \le R_1 + R_2, \end{cases}\end{aligned}$$

where $\gamma_i : [0, R_i] \to M$, is associative, and not only associative up to homotopy. In order for this topological category to induce an A_∞-category on the chains, we are going to construct the chains by using singular cubes.

Definition 8.1.1. *Fix $1 \le i \le k$. We call continuous maps $c : C^k \to M$ from the closed cube $[0, 1]^k$ to M singular k-cubes. Given such a singular k-cube C, we define the i-th* ***front face*** *of c to be the singular $(k-1)$-cube $F_i(c)$ given by:*

$$F_i(c)(x_1, \dots, x_{k-1}) := c(x_1, \dots, x_{i-1}, 0, x_i, \dots, x_{k-1}).$$

Similarly the i-th ***back face*** *is the singular $(k-1)$-cube $B_i(c)$ given by:*

$$B_i(c)(x_1, \dots, x_{k-1}) := c(x_1, \dots, x_{i-1}, 1, x_i, \dots, x_{k-1}).$$

We set the boundary of c to be:

$$\partial c := \sum_{i=1}^{k} (-1)^i (F_i c - B_i c).$$

Finally, a singular k-cube c is called degenerate if there exists some $1 \le i \le k$ such that c does not depend on x_i, i.e. it factors through C^{k-1}. We then set

$$C_i(X) := \frac{\mathbb{Z}[\text{singular cubes}]}{\mathbb{Z}[\text{degenerate cubes}].}$$

It is then easy to show that $(C_\bullet(X), \partial)$ forms a chain complex. This theory has the advantage over the ordinary definition of chains that there is a map

$$C_*(X) \times C_*(Y) \to C_*(X \times Y)$$

which is associative in an appropriate sense. In order to make this composition associative with respect to the sign convention used in Equation (7.1.1), we need to slightly adjust the definitions. Define the A_∞-category $\mathcal{P}(M)$ by letting its objects be points of M, and by setting the morphism spaces to be

$$\mathrm{Hom}_*(q^0, q^1) := C_{-*}(\Omega(q^0, q^1))$$

and the differential and composition

$$\mu_1^P \sigma := \partial\sigma, \tag{8.1.1}$$
$$\mu_2^P(\sigma_2, \sigma_1) := (-1)^{|\sigma_1|}\sigma_1 \cdot \sigma_2, \quad \text{where } |\sigma_1| = \deg(\sigma_1). \tag{8.1.2}$$

Writing out the associator, i.e. Equation (7.1.1) for $d = 3$, we obtain

$$\mu_2^P(\sigma_3, \mu_2^P(\sigma_2, \sigma_1)) + (-1)^{|\sigma_1|-1}\mu_2^P(\mu_2^P(\sigma_3, \sigma_2), \sigma_1) = \text{ some terms involving } \mu_1^P \text{ and } \mu_3^P.$$

Focusing on the LHS, we have

$$\begin{aligned}(-1)^{|\sigma_1|}\mu_2^P(\sigma_3, \sigma_1 \cdot \sigma_2) - (-1)^{|\sigma_1|+|\sigma_2|}\mu_2^P(\sigma_2 \cdot \sigma_3, \sigma_1) \\ = (-1)^{|\sigma_1|+|\sigma_1|+|\sigma_2|}\sigma_1 \cdot \sigma_2 \cdot \sigma_3 - (-1)^{|\sigma_1|+|\sigma_1|+|\sigma_2|}\sigma_1 \cdot \sigma_2 \cdot \sigma_3 \\ = 0.\end{aligned}$$

Thus we can set

$$\mu_d^P := 0 \tag{8.1.3}$$

for all $d \geq 3$. We will need to consider an enlargement of $\mathcal{P}(M)$ to a triangulated A_∞-category. For this we will take its twisted complex, but due to Equation (8.1.3), it will take on a much simpler form. First, we allow shifts of all objects by arbitrary integers and then define the morphism spaces

$$\mathrm{Hom}_*(q^0[m_0], q^1[m_1]) := \mathrm{Hom}_*(q^0, q^1)[m_1 - m_0]$$

with differential μ_1^P. Given a triple $(q^0[m_0], q^1[m_1], q^2[m_2])$, and morphisms $\sigma_i \in \mathrm{Hom}_*(q^{i-1}, q^i)$, we define the composition by

$$\mu_2^P(\sigma_2[m_2 - m_1], \sigma_1[m_2 - m_0]) = (-1)^{(|\sigma_2|+1)(m_1-m_0)}\mu_2^P(\sigma_2, \sigma_1)[m_2 - m_0]. \tag{8.1.4}$$

We will now restate Definition 7.3.8 in the special case where (8.1.3) holds.

Definition 8.1.2. *In the case specified above, a twisted complex consists of the following data:*

- *a finite collection of objects $\{q^i\}_{i=1}^r$ and integers m_i,*
- *a collection of morphisms $\{\delta_{ij}\}_{i<j}$ of degree 1 in $\mathrm{Hom}_*(q^i[m_i], q^j[m_j])$ such that*

$$\mu_1^P \delta_{ij} + \sum_k \mu_2^P(\delta_{kj}, \delta ik) = 0. \tag{8.1.5}$$

In this special setting, the data that goes into defining a twisted complex can be neatly packaged by denoting D for the matrix of morphisms $\{\delta_{ij}\}_{i<j}$ and by setting $T = (\oplus q^i[m_i], D)$ to be the twisted complex. Equation (8.1.5) can then be rewritten as

$$\mu_1^P(D) + \mu_2^P(D, D) = 0. \tag{8.1.6}$$

Given two such complexes $T^1 = (\oplus q_1^i[m_i^1], D^1)$ and $T^2 = (\oplus q_2^i[m_i^2], D^2)$, we set the space of morphisms between them as the direct sum

$$\mathrm{Hom}_*(T^1, T^2) := \oplus_{i_1,i_2} C_*(\Omega_{q_1^{i_1},q_2^{i_2}})[m_{i_1}^1 - m_{i_2}^2].$$

The differential of an element $S = \{\sigma_{i_1,i_2}\}$ in this space is given for an elementary matrix by

$$\sigma_{i_1,i_2} \mapsto \mu_1^P \sigma_{i_1,i_2} + \sum_{k<i_1} \mu_2^P(\sigma_{i_1,i_2}, \delta_{k,i_1}^1) + \sum_{i_2<k} \mu_2^P(\delta_{i_2,k}^2, \sigma_{i_1,i_2}) \tag{8.1.7}$$

which can be rewritten in terms of matrix multiplication

$$\mu_1^{\mathrm{Tw}(\mathcal{P})} S = \mu_1^P S + \mu_2^P(S, D^1) + \mu_2^P(D^2, S). \tag{8.1.8}$$

Composition is then given by

$$\mu_2^{\mathrm{Tw}(\mathcal{P})}(S_2, S_1) = \mu_2^P(S_2, S_1). \tag{8.1.9}$$

8.2 Wrapped Fukaya Category

Rather than immediately defining the wrapped Fukaya category on the cotangent bundle, we will define it for a more general class of manifolds.

Definition 8.2.1. *Let V be a symplectic manifold equipped with a 1-form λ whose differential ω is a symplectic form. Assume the existence of a compact (codimension 0) submanifold V^{in} with boundary such that the restriction of λ to ∂V^{in} is a contact form. We say that V is a* ***Liouville manifold*** *if we have a decomposition*

$$V = V^{\mathrm{in}} \cup_{\partial V^{\mathrm{in}}} [1, +\infty) \times \partial V^{\mathrm{in}},$$

such that the Liouville form is given by $\lambda = r \cdot \lambda|_{\partial M}$ on the infinite end $[1, +\infty) \times \partial V^{\mathrm{in}}$, where r is the coordinate on $[1, +\infty)$.

Instead of studying all Lagrangians in V, we consider a closed exact Lagrangian $M \subset V$, together with a finite collection $\mathrm{Obj}(\mathcal{W}_b(V))$ of exact properly embedded Lagrangians such that

(H6) λ vanishes on $L \cap (\partial V^{\mathrm{in}} \times [1, +\infty))$ if $L \in \mathrm{Obj}(\mathcal{W}_b(V))$, and M intersects each Lagrangian in $\mathrm{Obj}(\mathcal{W}_b(V))$ transversally.

Remark 8.2.2. The first condition is equivalent to the requirement that the intersection ∂L of L and ∂V^{in} is Legendrian, and that L decomposes as

$$L = L^{\mathrm{in}} \cup_{\partial L^{\mathrm{in}}} [1, +\infty) \times \partial L^{\mathrm{in}},$$

where $L^{\mathrm{in}} = V^{\mathrm{in}} \cap L$.

EXAMPLE 8.2.3. The example one should keep in mind is our eventual target manifold; the cotangent bundle. Recall that for a closed manifold M, the cotangent bundle with Liouville form $\lambda = \sum p_i dq_i$ decomposes as

$$T^*M = D^*M \cup_{S^*M} ([1, +\infty) \times S^*M),$$

with S^*M the contact boundary of D^*M.

Since the Lagrangians in $\mathrm{Obj}(\mathcal{W}_b(V))$ are exact, we are able to choose a primitive f_L for the restriction of λ to each such Lagrangian, which by (H6) means that f_L is locally constant away from a compact set. In addition, we choose a primitive f_M for $\lambda|_M$. The exactness condition will exclude the phenomena of disc bubbling when we later on look at compactifications of holomorphic discs.

These requirements are enough to define a $\mathbb{Z}_2$-graded Fukaya category. In order to also define it over the integers, additional assumptions are needed. Before we can state these, we need a cohomological definition of the Maslov index. The cohomology of the stable Lagrangian Grassmannian has the following isomorphism:

$$H^1(\mathrm{LGr}; \mathbb{Z}) \cong \mathbb{Z}.$$

Thus we are able to choose a generator, called the *universal Maslov index*

$$u \in H^1(\mathrm{LGr}; \mathbb{Z}).$$

Given any Lagrangian L, its tangent bundle is classified by a function

$$i : L \to \mathrm{LGr}$$

and the Maslov index of L is then defined as the pullback

$$\mu_L = i^*u \in H^1(L; \mathbb{Z}).$$

We can now pose the additional restrictions that will make the subsequent constructions work.

(H7) Assume for each $L \in \mathrm{Obj}(\mathcal{W}_b(V))$ or $L = M$ that: i) the restriction of b to L agrees with the second Stiefel-Whitney class $w_2(L)$; and ii) that the relative first Chern class $2c_1(V, L)$ vanishes on $H_2(V, L)$.

The reasons for this hypothesis are twofold. First of all, the condition on $c_1(V, L)$ enables us to put a grading on the Lagrangians: through the boundary map $H^1(L) \to H^2(M, L)$, we have that the Maslov class $\mu_L \in H^1(L)$ maps to the relative Chern class $2c_1(M, L)$. This together with (H7) implies that the Maslov class on $H_1(L)$ vanishes. The Maslov class is by definition the obstruction to consistently choose graded lifts of the tangent spaces to L, i.e. lift sections $L \to \mathrm{LGr}(TV),\ p \mapsto T_pL$ to sections of the infinite cyclic cover $\widetilde{\mathrm{LGr}}(TV)$[Aur13]. Equivalently, the vanishing of $2c_1(M, L)$, enables one to equip V with a quadratic complex volume form, when provided with any compatible almost complex structure. Since L is Lagrangian, we may then evaluate such a volume form to obtain a map $L \to \mathbb{C}^*$. Again, the Maslov index $H_1(L)$ then provides an obstruction to lifting this map to the universal cover

of $\mathbb{C}^*$. The choice of such a lift is then *called a grading of* L. For more details on this particular way to define the grading, we refer to [Sei08, Sec.(11j)].

Secondly, the condition on b is going to enable us to provide isomorphisms of orientation lines induced by future moduli spaces. The second Stiefel-Whitney class is precisely the obstruction to the existence of Pin-structures, see [Sei08, Sec(11i)]. The condition that $b|_L$ agrees with $w_2(L)$ allows one to choose a relative Pin-structure on M as well as objects of $\mathcal{W}_b(V)$ by fixing a triangulation of V so that M and the elements of $\mathrm{Obj}(\mathcal{W}_b(V))$ are subcomplexes and by fixing an orientable vector bundle E_b on the 3-skeleton of this triangulation whose second Stiefel-Whitney class is the restriction of b to the 3-skeleton. Hypothesis (H7), together with the fact that E_b is orientable implies that

$$w_2(TL|_{L_{[3]}} \oplus E_b|_{L_{[3]}}) = 0,$$

where $L_{[3]}$ stands for the 3-skeleton of L with respect to the triangulation induced by virtue of L being a subcomplex of V.

Definition 8.2.4. *A* ***relative*** Pin ***structure*** *on* L *is a choice of a* Pin *structure on the vector bundle* $TL|_{L_{[3]}} \oplus E_b|_{L_{[3]}}$ *defined over the 3-skeleton of* L.

A ***brane structure*** *on a Lagrangian is the choice of a grading, together with a relative* Pin *structure.*

From here on out, all of our considerations will be made with respect to some fixed brane structure on M and each object $L \in \mathrm{Obj}(\mathcal{W}_b(V))$.

8.2.a) Morphism Spaces of the Wrapped Fukaya Category

Having defined the objects of $\mathcal{W}_b(V)$, we now want to describe the morphisms. For this we will need to understand Hamiltonian chords linking two exact Lagrangians. The Liouville vector field is given by the equation

$$Z_\lambda := \iota_{Z_\lambda}\omega = \lambda,$$

and it agrees with $-r\partial_r$ on the cylindrical end of V. By Cartan's magic formula, we have that $\exp(tZ_\lambda)_*\omega = e^t\omega$ whenever the flow is defined, and thus intuitively one can think of the flow of the Liouville vector field as rescaling the symplectic form. We will denote by ψ^ρ the image of the negative Liouville flow for time $\log(\rho)$. Since we know the explicit formula on the cylindrical end of V, we have that

$$\psi^\rho(r, m) = (\rho \cdot r, m)$$

on it. Let $\mathscr{H}(V) \subset C^\infty(V, \mathbb{R})$ be the set of smooth functions H satisfying

$$H(r, y) = r^2$$

away from a compact subset of V and we write $\mathscr{H}_M(V)$ for the set of those $h \in \mathscr{H}(V)$ such that $H|_M = 0$. These functions serve as Hamiltonians on V and as usual, we define the Hamiltonian vector field X_H by

$$\iota_{X_H}\omega = dH.$$

Recall that in Definition 6.1.1, we defined Hamiltonian chords $x : [0,1] \to V$, and denoted the set of them by $\mathfrak{X}_H(L_0, L_1)$ for each pair $L_0, L_1 \in \mathrm{Obj}(\mathcal{W}_b(V))$. As before, by performing generic Hamiltonian perturbations of the Lagrangians in $\mathrm{Obj}(\mathcal{W}_b(V))$, we can assume that

(H8) all time-1 Hamiltonian chords of H with boundaries L_0 and L_1 are non-degenerate.

To each Hamiltonian chord $x \in \mathfrak{X}_H(L_0, L_1)$, we assign a Maslov index $|x|$ coming from the gradings on L_0 and L_1. This index $|x|$ corresponds to the Maslov index of the intersection $\Phi(L_0) \cap L_1$, where Φ denotes the time-1 Hamiltonian flow, see [Sei08, Sec.(11h].

We can now define the action of a chord x starting on L_0 and ending on L_1 similarly to Definition 6.1.4, with the key difference that the primitives might not be 0:

$$\mathcal{A}(x) := -\int_0^1 x^*(\lambda) + \int H(x(t))dt + f_{L_1}(x(1)) - f_{L_0}(x(0)). \tag{8.2.1}$$

The assumption that $H(r, m) = r^2$ away from a compact set implies that any chord that intersects some slice $\{r\} \times \partial V^{\mathrm{in}}$ is entirely contained therein and has action:

$$\mathcal{A}(x) = -r^2 + f_{L_1}(x(1)) - f_{L_0}(x(0)). \tag{8.2.2}$$

Lemma 8.2.5 ([Abo09, Lem. 3.1.]). *$\mathcal{A}$ is a proper map from $\mathfrak{X}_H(L_0, L_1)$ to $\mathbb{R}$.*

Proof. Non-degeneracy implies that there are only finitely many chords in any compact subset of M, while Equation 8.2.2 implies that the action of a sequence of chords which escapes every compact set must go to $-\infty$. □

Next, we set up the moduli space of strips. As in Section 6.3.a), fix a smooth map

$$\tau : [0,1] \to [0,1]$$

such that τ is identically 0 near 0 and 1 near 1. Given a pair of chords $x_0, x_1 \in \mathfrak{X}_H(L_0, L_1)$, define $\widetilde{\mathcal{R}}(x_0; x_1)$ to be the moduli space of maps

$$u : \mathbb{R} \times [0,1] \to V$$

satisfying:

i) $u(\mathbb{R} \times \{1\}) \subset L_1$,

ii) $u(\mathbb{R} \times \{0\}) \subset L_0$,

iii) $\lim_{s \to -\infty} u(s, \cdot) = x_0(\cdot)$,

iv) $\lim_{s \to +\infty} u(s, \cdot) = x_1(\cdot)$,

which solve

$$(Du - X_H \otimes d\tau)^{0,1} = 0. \tag{8.2.3}$$

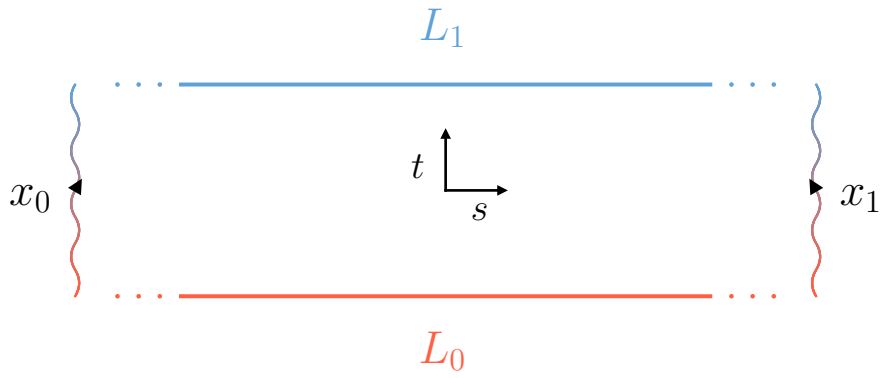


Figure II.1: Holomorphic strip. We indicate convergence to the chords x_0 and x_1.

By standard results, we have:

LEMMA 8.2.6 ([Abo09, Lem. 3.2.]). *The moduli space $\widetilde{\mathcal{R}}(x_0; x_1)$ is regular for a generic choice of almost complex structure I_t, and it has dimension $|x_0| - |x_1|$.*

Since Equation (8.2.3) is independent of s, the space $\widetilde{\mathcal{R}}(x_0; x_1)$ presents a natural $\mathbb{R}$-action by translation. We denote by $\mathcal{R}(x_0; x_1)$ the quotient of $\widetilde{\mathcal{R}}(x_0; x_1)$ by the $\mathbb{R}$-action whenever it is free, and set it to be 0 otherwise.

This space presents a natural compactification by adding broken strips to it. As in the Hamiltonian Floer case, the boundary strata of this new space $\overline{\mathcal{R}}(x_0; x_1)$ are then given by

$$\bigsqcup_{y_i} \mathcal{R}(x_0; y_1) \times \mathcal{R}(y_1; y_2) \times \cdots \times \mathcal{R}(y_{k+1}; x_1). \tag{8.2.4}$$

However it is not immediately clear that this space is actually compact since L_0 and L_1 trail of to infinity. The proof of the following result is similar to the ones of Lemma 2.3.5 and Lemma 2.3.7.

LEMMA 8.2.7 ([Abo09, Lem. 3.3.]). *If a solution to Equation (8.2.3) converges to x_0 at $-\infty$ and to x_1 at $+\infty$, then*

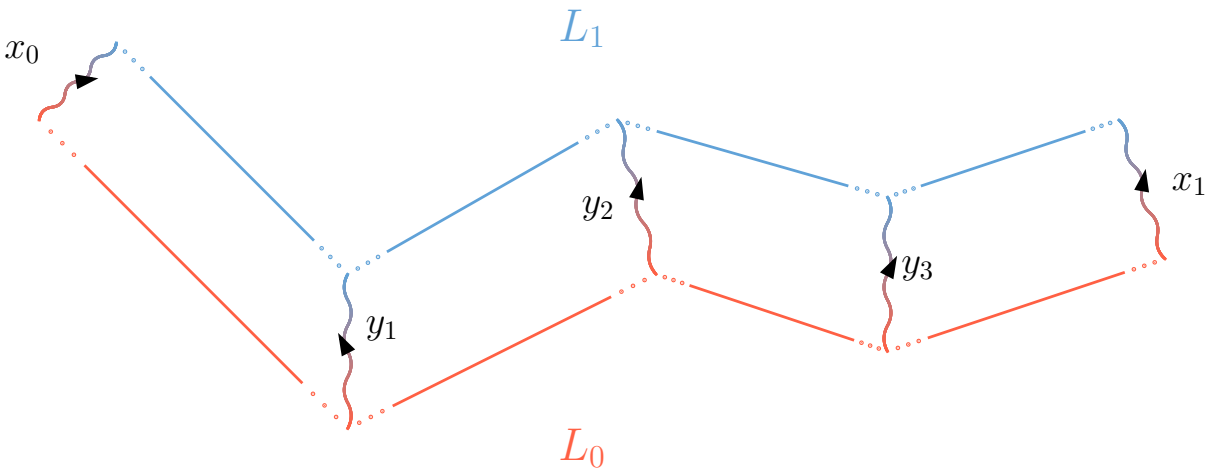
$$\mathcal{A}(x_0) \geq \mathcal{A}(x_1).$$

In addition, all such solutions lie entirely within a compact subset of V depending only on x_0 and x_1. $\square_{\text{ref.}}$

We can now apply the usual Gromov compactification to the moduli space $\mathcal{R}(x_0; x_1)$:

COROLLARY 8.2.8 ([Abo09, Cor. 3.4.]). *For each chord x_1, the moduli space $\overline{\mathcal{R}}(x_0; x_1)$ is empty for all but finitely many choices of x_0, and it is a compact manifold with boundary of dimension $|x_0| - |x_1| - 1$ whenever I_t is a generic family of almost complex structures. Moreover, the boundary is covered by the closure of the images of the natural inclusions*

$$\mathcal{R}(x_0; y) \times \mathcal{R}(y; x_1) \to \overline{\mathcal{R}}(x_0; x_1).$$



In order to define the Floer chain complexes which will serve as morphism spaces in the wrapped Fukaya category, we need to define a determinant bundle for each Hamiltonian chord y. Let T^- be the genus 0 Riemannian surface obtained by removing a point from the boundary of D^2, and equipping it with a negative end near the puncture. Note that T^- has a lot of similarities to T^+ of Equation (6.3.2), but it does not have punctures at the corners. Choose a function $[0, 1] \to [0, 1]$ which is locally constant near the boundary and define

$$\tau^- : T^- \to [0, 1]$$

to be the composition of this function with the projection. Let β^- be a 1-form on T^- which vanished near the boundary and agrees with the pullback of $d\tau^-$ along the negative end. We define the Cauchy-Riemann equation

$$(Du - X_H \otimes \beta)^{0,1} = 0 \tag{8.2.5}$$

on section of $T^- \times V$. The linearisation of this operator has as source sections of $(y \circ \tau^-)^*(TV)$. In addition, the gradings of L_0 and L_1 define a path of Lagrangians Λ_y in the restriction of this symplectic vector bundle to the boundary, that is uniquely determined up to homotopy by the requirement that it lifts to a path of graded Lagrangians interpolating between the graded structures on $T_{y(0)}L_0$ and $T_{y(1)}L_1$ on both sides of the puncture.

Definition 8.2.9. *The vector space o_y is the determinant bundle of the linearisation of Equation (8.2.5) with boundary conditions Λ_y.*

As usual, we then define $|o_y|$ to be generated by the two possible orientations of the rank 1 vector space o_y. We can now define the space that will eventually serve as a morphism space for the wrapped Fukaya category:

$$\mathrm{CW}_b^*(L_0, L_1) := \bigoplus_{x \in \mathfrak{X}_H(L_0, L_1)} |o_x|.$$

8.2.b) The Differential and Composition

In this section, we are going to define the differential and composition for the wrapped Fukaya category. Having defined the Floer complex through orientation lines, a lot of the following results immediately follow from similar techniques to those utilised in Chapter I. Therefore we are subsequently going to state a few of the results without proof.

Given two Hamiltonian chords $x_0, x_1 \in \mathfrak{X}_H(L_0, L_1)$ such that $|x_1| = |x_0| + 1$, we have by our assumptions that every element of u of the moduli space $\mathcal{R}(x_0; x_1)$ is rigid. By a construction similar to Lemma 2.5.4, that is presented in [Abo09, Sec. A.1.], we have that there is an isomorphism

$$o_{x_1} \to o_{x_0}. \tag{8.2.6}$$

Note that in order to define such isomorphisms in a way that is compatible with breaking, one needs the existence of relative Pin-structures, which are guaranteed by (H7). We denote the associated map on orientation lines by μ_u, and define the differential to be:

$$\begin{aligned} \mu_1^F : \mathrm{CW}_b^i(L_0, L_1) &\to \mathrm{CW}_b^{i+1}(L_0, L_1) \\ [x_1] &\mapsto (-1)^i \sum_{\substack{x_0 \in \mathfrak{X}_H(L_0, L_1) \\ |x_1| = |x_0| + 1}} \sum_{u \in \mathcal{R}(x_0; x_1)} \mu_u([x_1]). \end{aligned} \tag{8.2.7}$$

The proof that $\mu_1^F \circ \mu_1^F = 0$ is similar to the one of Proposition 2.5.9, where the finiteness of the sum on the RHS of (8.2.7) is guaranteed by Corollary 8.2.8. That is why we are only going to give some geometric intuition as to why $(\mu_u^F)^2 = 0$; after all, why should a composition of strips be zero? However, we are going to slightly postpone this discussion to Section 9.2, as this issue can be discussed more concisely once the general setup for the A_∞-structures has been put in place.

Even though Floer complexes are unaffected by generic perturbations in their Floer data, it will turn out that higher products are sometimes able to detect such perturbations. Whenever it will be important to distinguish between the Floer data, we will denote the Floer complexes by

$$\mathrm{CW}_b^*(L_0, L_1; \omega, H, I_t)$$

to be more precise.[4]

Lemma 8.2.10 ([Abo09, Lem. 3.5.]). *If $\psi : V \to V$ satisfies $\psi^*(\omega) = \rho\omega$ for some non-zero constant ρ, then we have a canonical isomorphism*

$$\mathrm{CW}(\psi) : \mathrm{CW}_b^*(L_0, L_1; \omega, H, I_t) \cong \mathrm{CW}_b^*(\psi(L_0), \psi(L_1); \omega, \frac{H}{\rho} \circ \psi, \psi^* I_t). \tag{8.2.8}$$

Proof. Given two Hamiltonian chords $x_0, x_1 \in \mathfrak{X}_H(L_0, L_1)$ and a solution $u \in \mathcal{R}(x_0; x_1)$, we obtain an analogous solution for the symplectic form $\psi^*\omega$ and families of almost complex structure I_t by considering the composition $\psi \circ u$. Thus we obtain an isomorphism of chain complexes

$$\mathrm{CW}(\psi) : \mathrm{CW}_b^*(L_0, L_1; \omega, H, I_t) \cong \mathrm{CW}_b^*(\psi(L_0), \psi(L_1); \psi^*\omega, H \circ \psi, \psi^* I_t). \tag{8.2.9}$$

The key observation is now that the condition $\psi^*\omega$ implies that the Hamiltonian flow of $H \circ \psi$ with respect to $\psi^*\omega$ agrees with the flow of $\frac{H}{\rho} \circ \psi$ with respect to ω; indeed, choose $Y \in \mathfrak{X}(V)$, and let us temporarily denote the Hamiltonian vector field of H with respect to ω by X_H^ω. Then:

$$\begin{aligned} \psi^*(\omega)(X_{H\circ\psi}^{\psi^*(\omega)}, Y) &= d(H \circ \psi) \\ \Rightarrow \rho\omega(X_{H\circ\psi}^{\psi^*\omega}, Y) &= d(H \circ \psi \\ \Rightarrow \ \omega(X_{H\circ\psi}^{\psi^*\omega}, Y) &= d(\frac{H}{\rho} \circ \psi). \end{aligned}$$

Since Y was arbitrary, we conclude that $X_{H\circ\psi}^{\psi^*\omega} = X_{\frac{H}{\rho}\circ\psi}^{\omega}$. Thus we obtain an isomorphism

$$\mathrm{CW}_b^*(\psi(L_0), \psi(L_1); \psi^*\omega, H \circ \psi, \psi^* I_t) \cong \mathrm{CW}_b^*(\psi(L_0), \psi(L_1); \omega, \frac{H}{\rho} \circ \psi, \psi^* I_t).$$

□

We set

$$\mathrm{CW}_b^*(\psi(L_0, \psi(L_1)) := \mathrm{CW}_b^*(\psi(L_0), \psi(L_1); \omega, \frac{H}{\rho} \circ \psi, \psi^* I_t),$$

and define ψ^2 to be the time-log(2) negative Liouville flow. Given three Lagrangians L_0, L_1 and L_2, we will eventually define a map

$$\mu_2^{\psi^2} : \mathrm{CW}_b^*(L_1, L_2) \otimes \mathrm{CW}_b^*(L_0, L_1) \to \mathrm{CW}_b^*(\psi^2(L_0), \psi^2(L_1)). \tag{8.2.10}$$

[4] As for any self-respecting mathematician we will, however, almost always suppress this extra data from the notation.

After composition with $\mathrm{CW}(\psi^2)^{-1}$, we then have the product

$$\mu_2^F : \mathrm{CW}_b^*(L_1, L_2) \otimes \mathrm{CW}_b^*(L_0, L_1) \to \mathrm{CW}_b^*(L_0, L_2)$$

in the wrapped Fukaya category.

The map (8.2.10) will count the equivalent of holomorphic pairs of pants in the Lagrangian setting. For this, let S be the Riemannian surface obtained by removing 3 points (ξ^0, ξ^1, ξ^2) from the boundary of D^2. We are then going to count solutions to a Cauchy-Riemann equation

$$(Du - X_S \otimes \alpha_S)^{0,1} = 0, \tag{8.2.11}$$

where α_S is a closed 1-form on S, and X_S is the Hamiltonian vector field of a function H_S on V that is parametrised by points of S.

Furthermore, we require that for a point S, the Hamiltonian $H_S \in \mathscr{H}(V)$ and restricts to H near $\mathscr{H}(V)$ and restricts to H near ξ^1 and ξ^2, in addition to restricting to $\frac{H}{4} \circ \psi^2$ near ξ^0. This makes sense since $\frac{H}{4} \circ \psi^2$ lies in $\mathscr{H}(V)$ since H agrees with r^2 on the cylindrical end. We will need the shift on the Hamiltonian near ξ^0 to account for the shift in Equation (8.2.10). To further specify the Floer data, we equip ξ^0 with a negative end, and ξ^1 and ξ^2 with a positive end:

$$\begin{aligned} &\varepsilon^0 : Z^- \to S \\ &\varepsilon^i : Z^+ \to S \quad \text{for } i = 1, 2 \end{aligned}$$

which map $\partial Z^{\pm}$ to ∂S and converge to the respective puncture points ξ^i. The closed 1-form α_S is required to vanish on ∂S, and respect

$$\begin{aligned} (\varepsilon^0)^*(\alpha_S) &= 2d\tau \\ (\varepsilon^i)^*(\alpha_S) &= d\tau \quad \text{for } i = 1, 2. \end{aligned}$$

In addition, we choose a family of almost complex structures

$$I_S : S \to \mathcal{J}(V)$$

parametrised by points of S, whose composition with ε^1 and ε^2 agrees with I_t and the composition with ε^0 agrees with $(\psi^2)^* I_t$.

Similar to previous constructions, we want that solutions to (8.2.11) have Lagrangian boundary conditions (L_0, L_1) near ξ^1, (L_1, L_2) near ξ^2, and $(\psi^2(L_0), \psi^2(L_2))$ near ξ^0. In contrary to the other Lagrangian boundary conditions from before, this however implies that we cannot have constant boundary conditions, and that we need to interpolate between them. In order to achieve this, we make a choice of a map $\rho_S : \partial D^2 \to [1, 2]$ such that:

$$\begin{cases} \rho_S(z) = 1 & \text{near } \xi^1 \text{ and } \xi^2 \\ \rho_S(z) = 2 & \text{near } \xi^0. \end{cases}$$

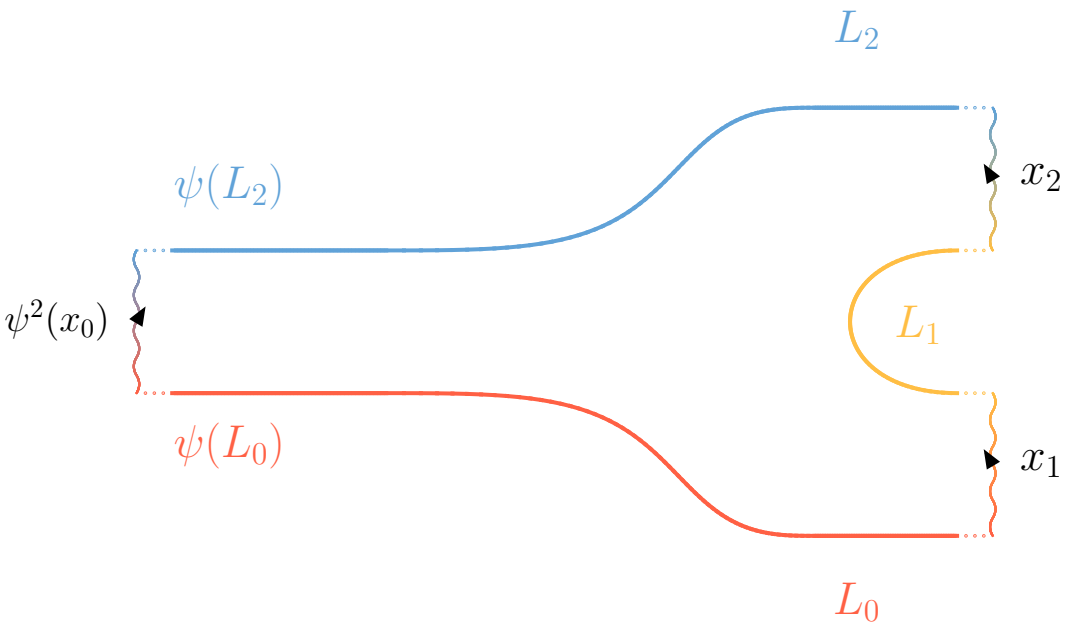
Definition 8.2.11. *The moduli space $\mathcal{R}_2(x_0; x_1, x_2)$ is the space of solutions u to Equation (8.2.11) with boundary conditions:*

$$\begin{cases} u(z) \in \psi^{\rho_S(z)}(L_1) & \text{if } z \in \partial S \text{ lies between } \psi_1 \text{ and } \psi_2, \\ u(z) \in \psi^{\rho_S(z)}(L_2) & \text{if } z \in \partial S \text{ lies between } \psi_2 \text{ and } \psi_0, \\ u(z) \in \psi^{\rho_S(z)}(L_0) & \text{if } z \in \partial S \text{ lies between } \psi_0 \text{ and } \psi_1. \end{cases}$$

Furthermore, the image of u should converge to x_1 and x_2 on the strip-like ends ε^1 and ε^2 respectively, and to $\psi^2(x_0)$ on ε^0.

The above construction ensures that the pullback of Equation (8.2.11) under the strip-like ends ξ^i are given by

$$\begin{cases} (Du - X_H \otimes d\tau)^{0,1} = 0 & \text{w.r.t. } I_t \text{ for } i = 1, 2, \text{ and,} \\ (Du - 2X_{\frac{H}{4} \circ \psi} \otimes d\tau)^{0,1} = 0 & \text{w.r.t. } (\psi^2)^* I_t \text{ for } i = 0. \end{cases}$$



Thus for the inputs x_1 and x_2, we exactly recover Equation (8.2.3), while for the output we recover the same type of equation, but for the Hamiltonian $\frac{H}{2} \circ \psi^2$. The Gromov bordification is thus obtained by adding strata that contain broken strips

at the respective punctures ξ^i.It is then obtained by adding

$$\begin{aligned}
&\bigsqcup_{y\in\mathfrak{X}_H(L_0,L_1)} \mathcal{R}_2(x_0;y,x_2)\times\overline{\mathcal{R}}(y;x_1)\\
&\bigsqcup_{y\in\mathfrak{X}_H(L_1,L_2)} \mathcal{R}_2(x_0;x_1,y)\times\overline{\mathcal{R}}(y;x_2)\\
&\bigsqcup_{y\in\mathfrak{X}_H(L_0,L_2)} \overline{\mathcal{R}}(x_0;y)\times\mathcal{R}_2(y;x_1,x_2),
\end{aligned} \tag{8.2.12}$$

to $\mathcal{R}_2(x_0;x_1,x_2)$. The factor $\overline{\mathcal{R}}(x_0;y)$ in the last stratum might be surprising since one would expect the moduli space $\overline{\mathcal{R}}(\psi^2(x_0),\psi^2(y))$ instead considering the moving Lagrangian boundary conditions; however both spaces can be identified by using $(\psi^2)^{-1}$.

The following lemma follows from a standard transversality argument, together with a compactness argument using the integrated maximum principle.

Lemma 8.2.12 ([Abo09, Lem. 3.7.]). *For a fixed pair (x_1,x_2), the moduli space $\mathcal{R}_2(x_0;x_1,x_2)$ is empty for all but finitely many choices of chords x_0. For a generic family of almost complex structures I_S and Hamiltonians H_S, $\overline{\mathcal{R}}_S(x_0;x_1,x_2)$ is a compact manifold of dimension*

$$|x_0|-|x_1|-|x_2|$$

whose boundary is covered by the codimension 1 strata of (8.2.12).

If $|x_0|=|x_1|+|x_2|$, then $\mathcal{R}_2(x_0;x_1,x_2)$ is of dimension 0 and consists of finitely many elements. Furthermore, a gluing argument presented in [Abo09, Eq.(A.4)] yields an isomorphism of orientation lines:

$$o_{x_2}\otimes o_{x_1}\to o_{\psi^2(x_0)}$$

whose existence again assumes the existence of a relative Pin-structure. Writing again μ_u for the above isomorphism induced by a solution $u\in\mathcal{R}_2(x_0;x_1,x_2)$, we defined the product on the wrapped Fukaya category by:

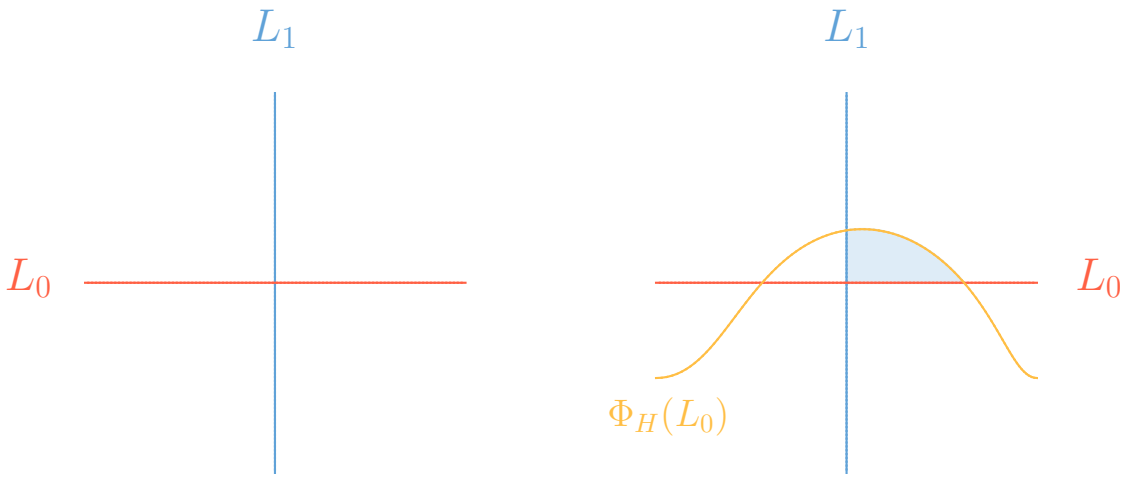
$$\begin{aligned}
&\mu_2^{\psi^2}:\mathrm{CW}_b^*(L_1,L_2)\otimes\mathrm{CW}_b^*(L_0,L_1)\to\mathrm{CW}_b^*(\psi^2(L_0),\psi^2(L_2))\\
&\qquad\mu_2^{\psi^2}([x_2],[x_1]):=\sum_{\substack{x_1\in\mathfrak{X}_H(L_0,L_1)\\ x_2\in\mathfrak{X}_H(L_1,L_2)\\ x_0\in\mathfrak{X}_H(\psi^2(L_0),\psi^2(L_2))\\ |x_0|=|x_1|+|x_2|}}\ \sum_{u\in\mathcal{R}_2(x_0;x_1,x_2)}(-1)^{|x_0|}\mu_u([x_2],[x_1]).
\end{aligned}$$

We can then concatenate the product with $\mathrm{CW}(\psi^2)^{-1}$ to obtain a map

$$\mu_2^F:\mathrm{CW}_b^*(L_1,L_2)\otimes\mathrm{CW}_b^*(L_0,L_1)\to\mathrm{CW}_b^*(L_0,L_2).$$

We can now make precise what we had in mind when we claimed that μ_2^F is sensitive to Hamiltonian perturbations. Consider a non-transverse case, e.g. $L_2=L_0$. Let Φ_H denote the time-1 Hamiltonian flow. Roughly speaking, μ_2^F counts holomorphic

triangles with Lagrangian boundary conditions L_0, L_1 and L_2. If we now perform a Hamiltonian perturbation on L_2, we have the following picture:



From this we can infer that a Hamiltonian perturbation might introduce a non-trivial μ_2^F map, even though the Floer complex can not detect them. It is thus of paramount importance that we chose our Lagrangians at the beginning such that they intersect transversally, and that we keep track of the perturbations we introduce while defining the A_∞-relations.

9 A_∞-Relations of the Wrapped Fukaya Category

We are now going to present the construction of higher products in the wrapped Fukaya category. As it turns out, they will arise from a careful analysis of the moduli space of punctured discs.

9.1 The Deligne-Mumford Compactification

In the upcoming section, we will present how the A_∞-relations of the wrapped Fukaya category are intimately linked to the combinatorial structure of the Stasheff associahedron.

We will denote by $\mathcal{R}_d$ the moduli space of abstract discs with one negative puncture, denoted by ξ^0, and d positive punctures denoted $\{\xi^i\}_{i=1}^d$, which are ordered in a clock-wise fashion. As it turns out, there is a compactification of this moduli space called the Deligne-Mumford compactification [DM69] of $\mathcal{R}_d$, and we will reference it by $\overline{\mathcal{R}}_d$. This compactification is for instance presented in [Sei08, Sec.(9f)]. We have the following important result that is proved therein.

THEOREM 9.1.1. *The Deligne-Mumford compactification $\overline{\mathcal{R}}_{d+1}$ is a compact manifold with corners, and it is diffeomorphic to the Stasheff associahedron K_d.*

Recall that the boundary faces of K_d were given by Stasheff associahedra of lower dimension. Thus $\overline{\mathcal{R}}_{d+1}$ also inherits such a decomposition.

Next, assume that a *universal and consistent* choice of strip-like ends has been made as defined in [Sei08, Sec.(9g)]. Explicitly, this means that for each surface S and each puncture ξ^i, we have a choice of maps

$$\varepsilon^i : Z^\pm \to S$$

that varies smoothly with the modulus of S in the interior of the moduli space. Furthermore, near each boundary stratum σ of $\overline{\mathcal{R}}_d$, the strip-like ends should be obtained by gluing: if we write $\mathcal{R}_{d_1} \times \cdots \times \mathcal{R}_{d_j}$, then gluing the strip-like ends chosen on these lower dimensional moduli spaces defined an embedding

$$\sigma \times [1, +\infty)^{j-1} \to \mathcal{R}_d\,.$$

A surface in the image of this gluing map is then by definition covered by patches coming from surfaces in one of the factors of the product decomposition of σ. In particular, each end of such a surface obtained by gluing comes with strip-like ends induced from the choices of strip-like ends on the lower dimensional moduli spaces. Consistency of the choice of strip-like ends is then the requirement that our choice on $\mathcal{R}_d$ agrees with this fixed choice in some neighbourhood σ.[5]

We can now define the Floer data to define a Cauchy-Riemann equation on these discs.

DEFINITION 9.1.2. *A Floer datum D_S on a stable disc $S \in \overline{\mathcal{R}}_d$ consists of the following choices:*

[5] [Abo09, p. 18.]

i) Weights: a positive integer $w_{i,S}$ assigned to the i-th end such that

$$w_{0,S} = \sum_{1 \leq i \leq d} w_{i,S}.$$

ii) Moving conditions: a map $\rho_S : \partial\overline{S} \to [1, +\infty)$ which agrees with w_i near the i-th end.

iii) Basic 1-form: a closed 1-form α_S whose restriction to the boundary vanishes and whose pullback under ε^i agrees with $w_{i,S} d\tau$.

iv) Hamiltonian perturbations: a map $H_S : S \to \mathscr{H}(V)$ which agrees with $\frac{H \circ \psi^{w_{i,s}}}{(w_{i,S})^2}$ near the i-th end.

v) Almost complex structure: a map $I_S : S \to \mathcal{J}(V)$ whose pullback under ε^i agrees with $(\psi^{w_{i,S}})^* I_t$.

The Floer data can be seen as the generalisation of the case $d = 2$ seen previously, and we set

$$(Du - X_S \otimes \alpha_S)^{0,1} = 0 \tag{9.1.1}$$

to be the Cauchy-Riemann equation with respect to the almost complex structure I_S, where X_S denotes the Hamiltonian flow of H_S. As always the main reason for the specific guise of the Floer data are that this equation restricts to desired Cauchy-Riemann equations at the strip-like ends. The following result is a straight-forward consequence of the definitions.

Lemma 9.1.3 ([Abo09, Lem. 3.9.]). *The pullback of Equation (9.1.1) under ε^i is given by*

$$(Du \circ \varepsilon^i - X_{\frac{H \circ \psi^{w_{i,S}}}{w_{i,S}}} \otimes d\tau) = 0. \tag{9.1.2}$$

In particular, it agrees with Equation (8.2.3) up to applying $\psi^{w_{i,S}}$.

9.2 Non-Associativity of the Product and $\mu_1^F \circ \mu_1^F = 0$

All of the A_∞-relations are going to be shown to be the result of certain degenerations at the boundary of the compactification of the moduli space of pseudo-holomorphic maps from $\overline{\mathcal{R}}_{d+1}$ into V. A lot of the intuition about breakings that occur in the moduli spaces of punctured discs was gained through the lectures of James Pascaleff [Pas20], who in turn drew from [Sei08] and [Aur13].

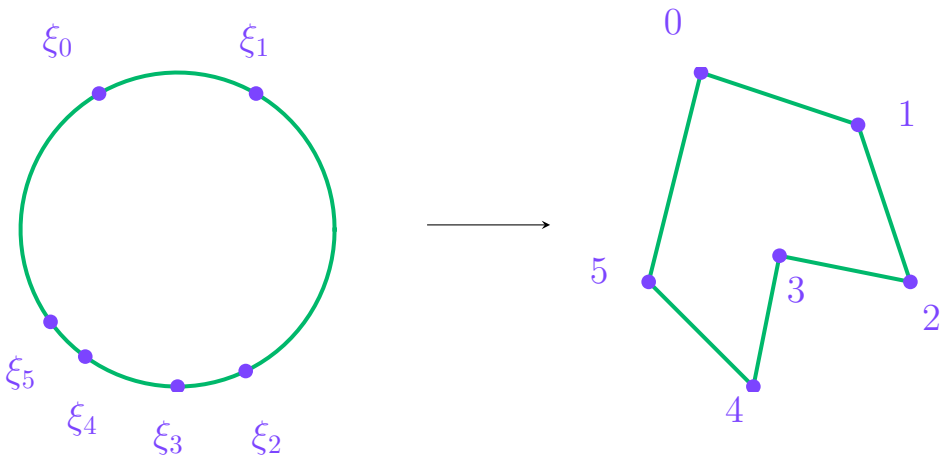
We are now going to illustrate this breaking at the boundary by first considering the cases $d = 1$ and $d = 2$. We will consider the Liouville manifold $\mathbb{C}$ together with its standard symplectic form and almost complex structure i; Lagrangians are then given by lines. We will use the following consequence of the Riemann mapping theorem.

THEOREM 9.2.1 (Schwarz–Christoffel mapping, [Dri02]). *Let P be a polygon in $\mathbb{C}$ with interior angles $\pi\alpha_k$ for $k \in \{1, \dots, n\}$, $\alpha_k \in (0, 2]$. The function $F(w)$ that maps $(D^2)^0$ biholomorphically onto P is explicitly given by*

$$F(w) = C \cdot \int_0^w \prod_{k=1}^{n} (w' - w_k)^{\alpha_k - 1} dw' + C',$$

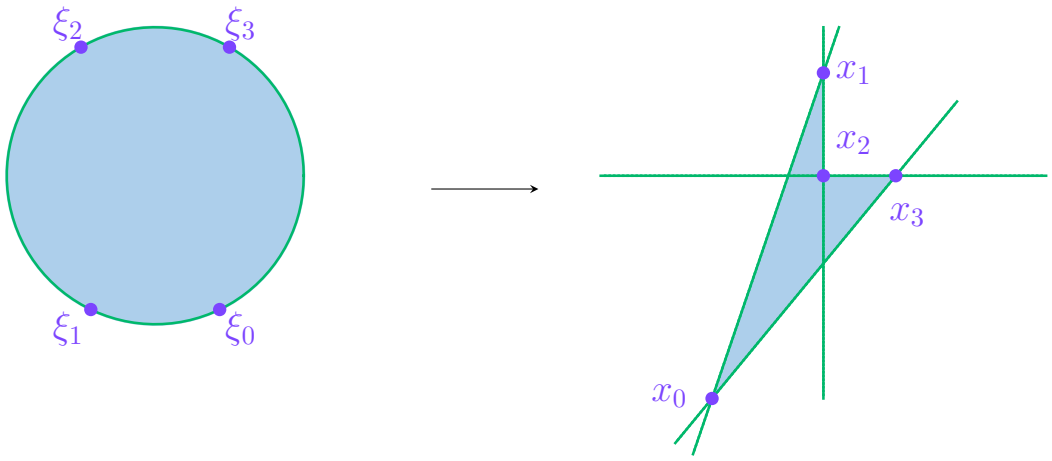
where w_k are the preimages of the edges of P under the Riemann mapping theorem.

This formula actually presents singularities at the points w_k, but this does not matter as we will always consider boundary punctured discs. We thus have an identification

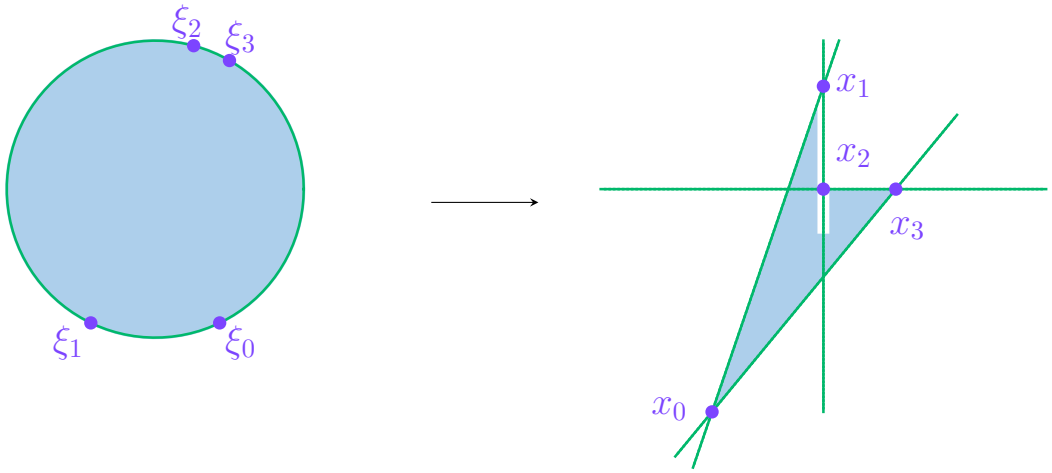


and we can represent any polygon in $\mathbb{C}$ by a punctured disc. This new point of view can help us understand the stratification of $\overline{\mathcal{R}}_{d+1}$. Consider for a example the following 4-gon[6]:

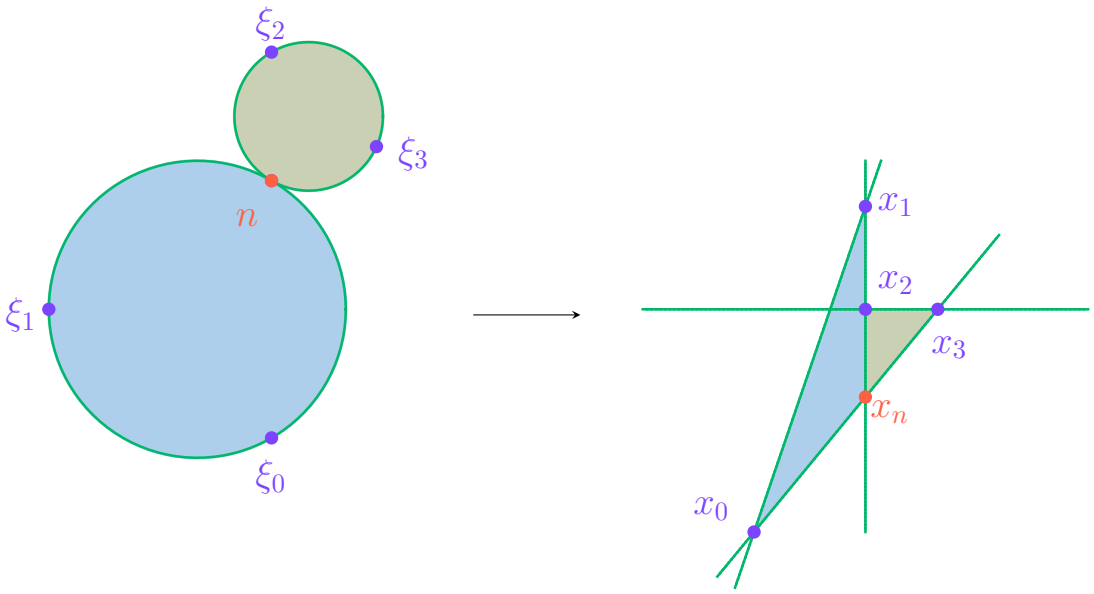
[6]Note that for all of our subsequent drawings, we will draw constant Hamiltonian chords corresponding to Lagrangian intersections. This restriction is done solely for the sake of presentation and does not influence the underlying ideas.



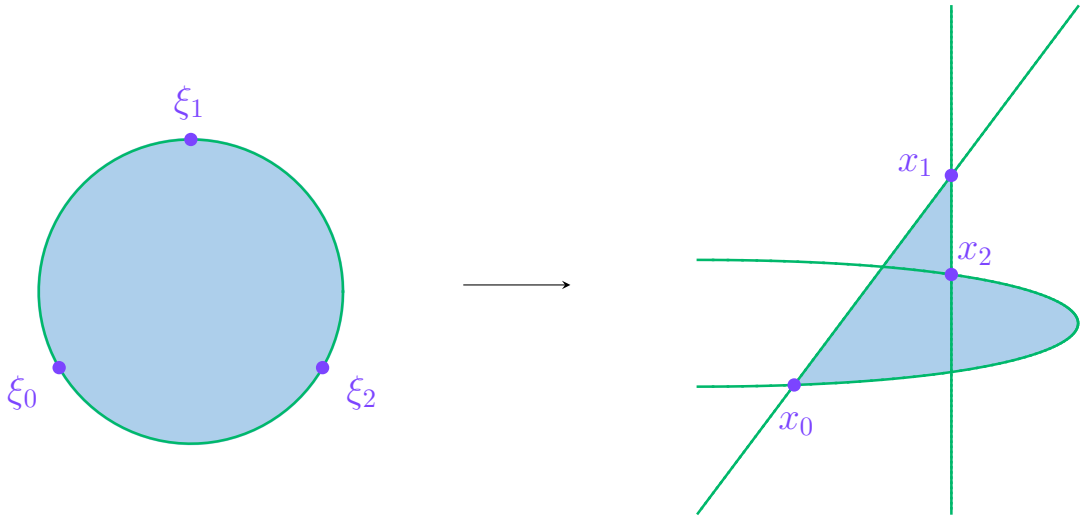
In the above picture the lines correspond to Lagrangians and the interior of the punctured disc gets mapped to the blue coloured 4-gon. At a corner with inner-angle bigger than π (in the above case at ξ_2), a slit can form along one of the intersecting Lagrangian. This happens for instance when we bring ξ_2 and ξ_3 closer together:



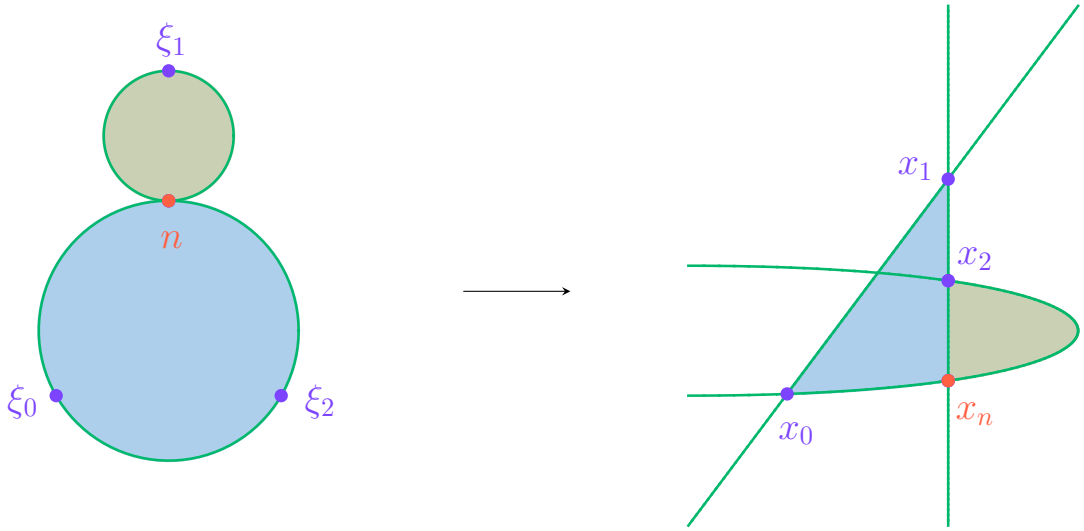
Eventually, as more and more energy accumulates around ξ_2 and ξ_3, we have that in the limit the above 4-gon breaks into two 3-gons that share a common edge.



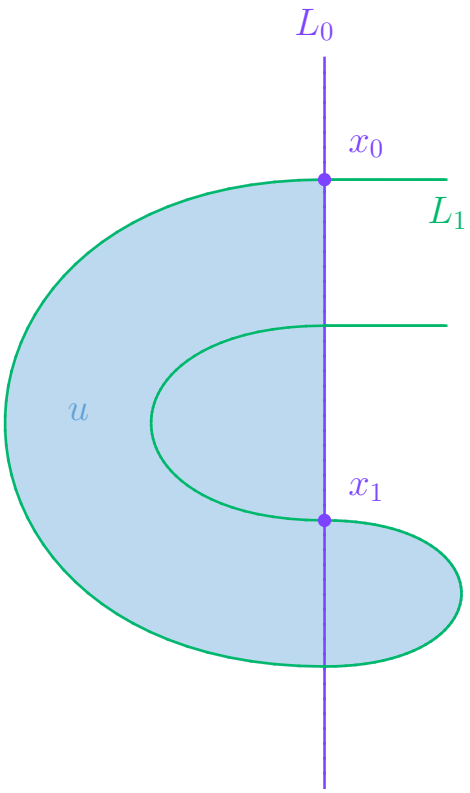
We will call such a breaking a *stable breaking*. Another situation that can occur is *strip breaking*, or *semi-stable breaking*:



Which after degeneration breaks into a 3-gon and a strip:


ξ_1
n
ξ_0
ξ_2
x_1
x_2
x_0
x_n

These two phenomena immediately imply that $\mu_1^F \circ \mu_1^F = 0$ and the possible failure to be associative of μ_2^F. For the first claim, consider a solution $u \in \mathcal{R}(x_0, x_1)$ for $|x_1| = |x_0| + 2$, and $x_0, x_1 \in \mathfrak{X}_H(L_0, L_1)$ given by the following picture:



There are then two ways in which a slit can form that eventually leads to a strip breaking off, once along L_0 and once along L_1:

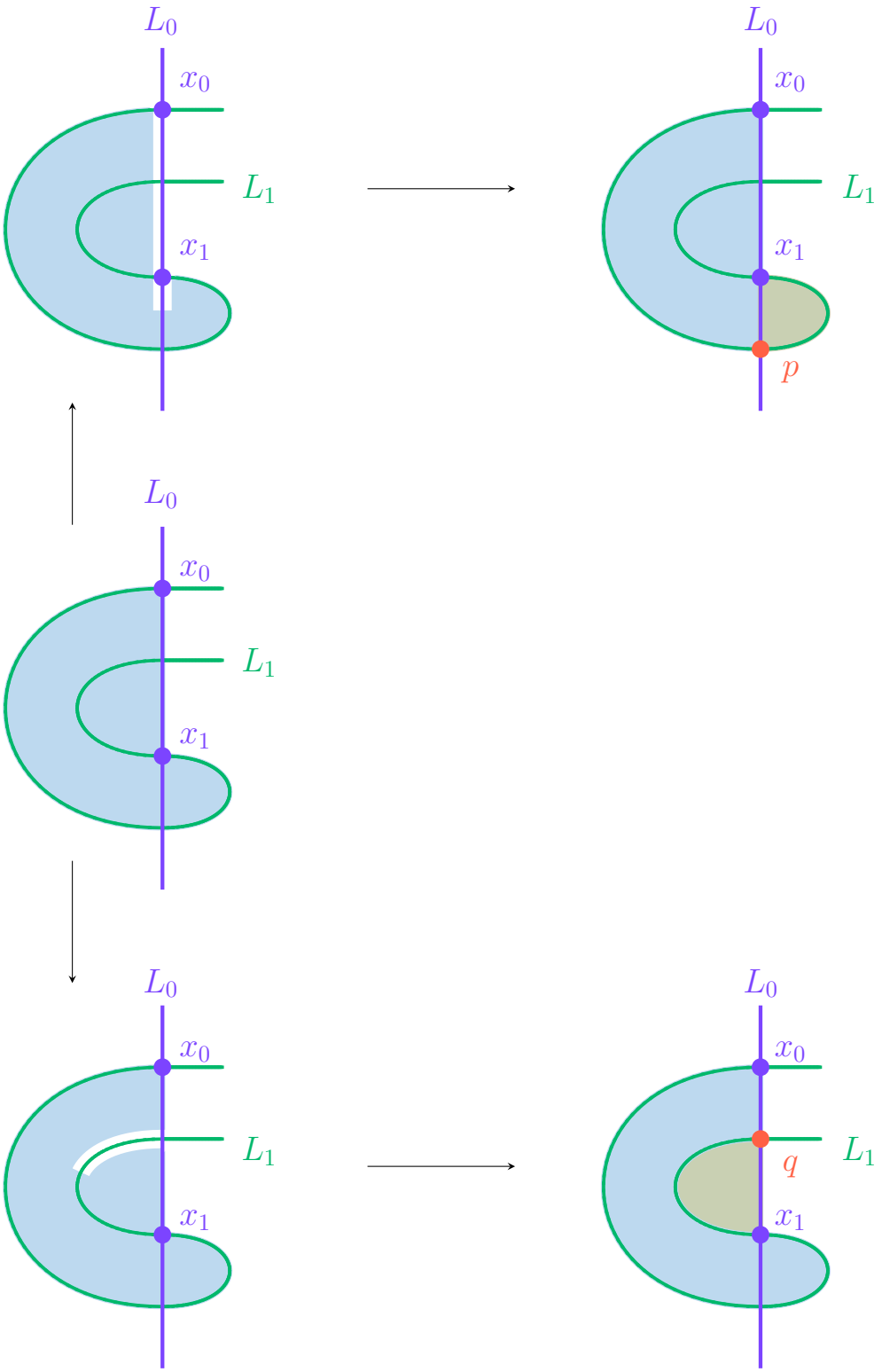


Figure II.2: The top row depicts a slit forming along L_0 that in the limit degenerates into a strip breaking off. The bottom row represents a slit forming along L_1.

The solution u thus provides a cobordism that associates two pairs of strips. Since $|x_1| = |x_0| + 2$, we have that $\mathcal{R}(x_0; x_1)$ is 1-dimensional and compact, thus the contributions of the strips that lie on the boundary of this cobordism, weighted by orientations, add to 0. One thing that one has to be wary of is the possibility of disc breaking, which is the equivalent of sphere bubbling in the Hamiltonian setting. This phenomena of holomorphic discs splitting off is however prohibited by our assumption that the symplectic form is exact.

The A_∞-relation for μ_2^F require both stable and semi-stable breakings. Later on, when we have explicitly defined the A_∞-product μ_3^F, the stable breakings will correspond to

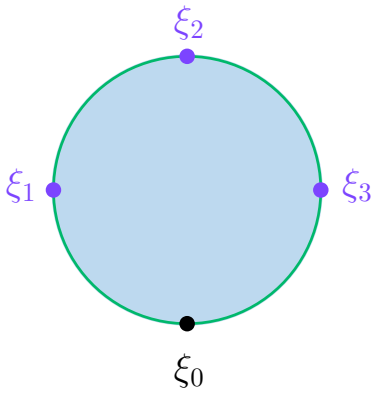
$$\mu_2^F(x_0; x_3, \mu_2^F(x_2, x_1)) \text{ and } \mu_2^F(x_0; \mu_2^F(x_3, x_2), x_1);$$

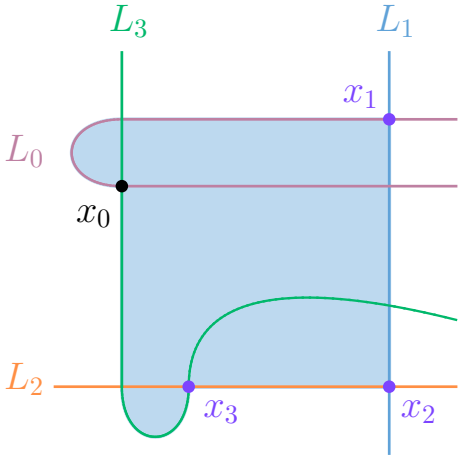
where $x_0 \in \mathfrak{X}_H(L_0, L_3)$ and $x_i \in \mathfrak{X}_H(L_{i-1}, L_i)$ for $i = 1, 2, 3$. The semi-stable breakings will correspond to

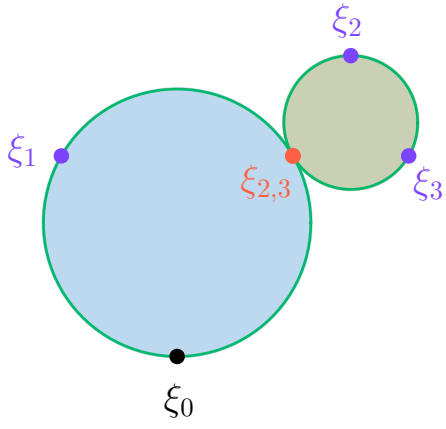
$$\mu_3^F(x_0; \mu_1^F(x_3), x_2, x_1),\ \mu_3^F(x_0; x_3, \mu_1^F(x_2), x_1),\ \mu_3^F(x_0; x_3, x_2, \mu_1^F(x_1))\ \text{, and}$$
$$\mu_1^F(\mu_3^F(x_0; x_3, x_2, x_1)).$$

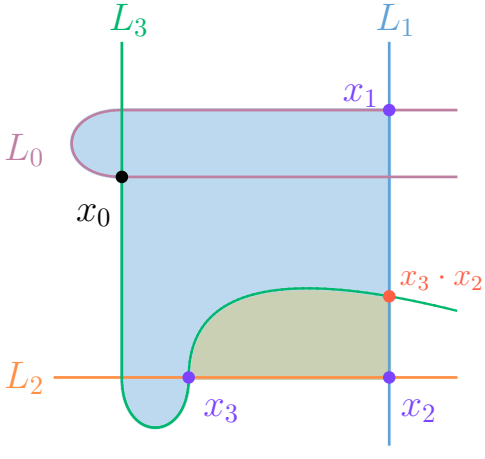
Here is a constellation of Lagrangians that exhibits all three forms of breakings:

(1)



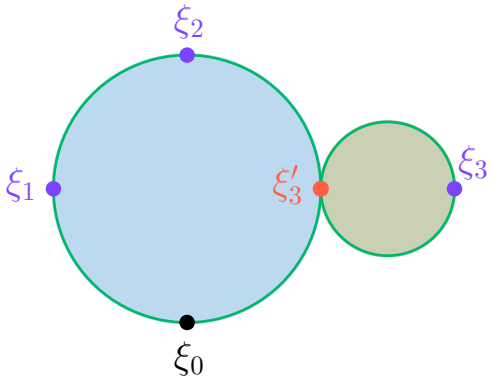


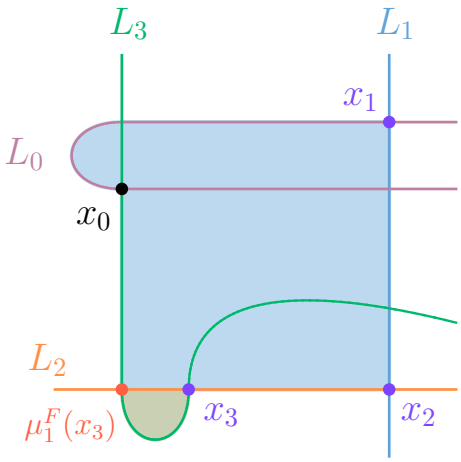




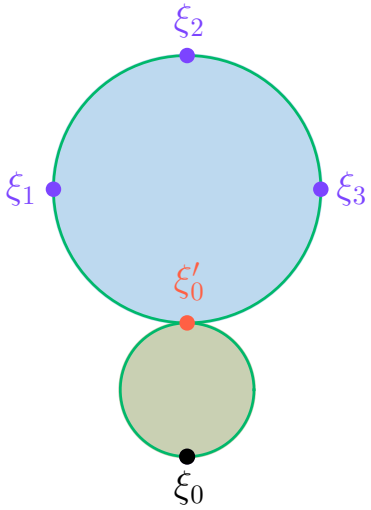
But there are also strip breakings that are possible:

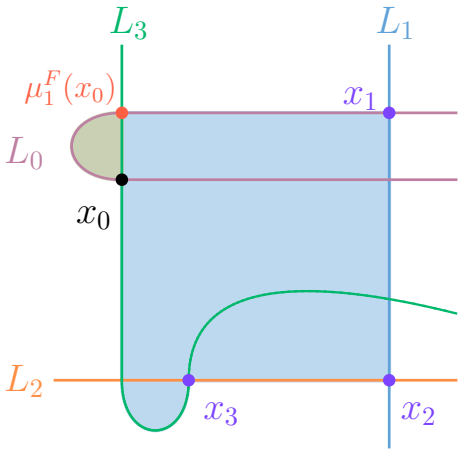
(2)





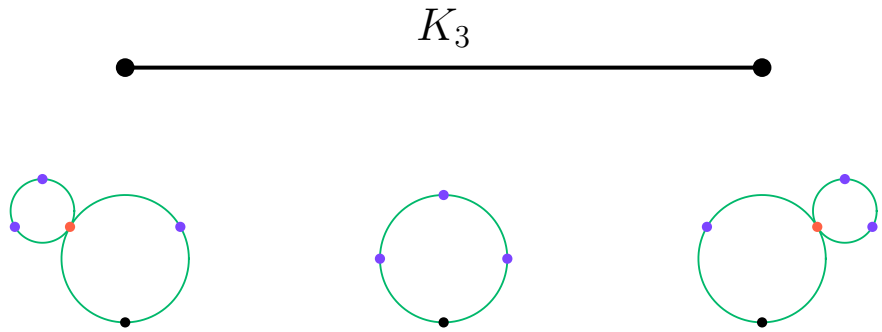
(3)



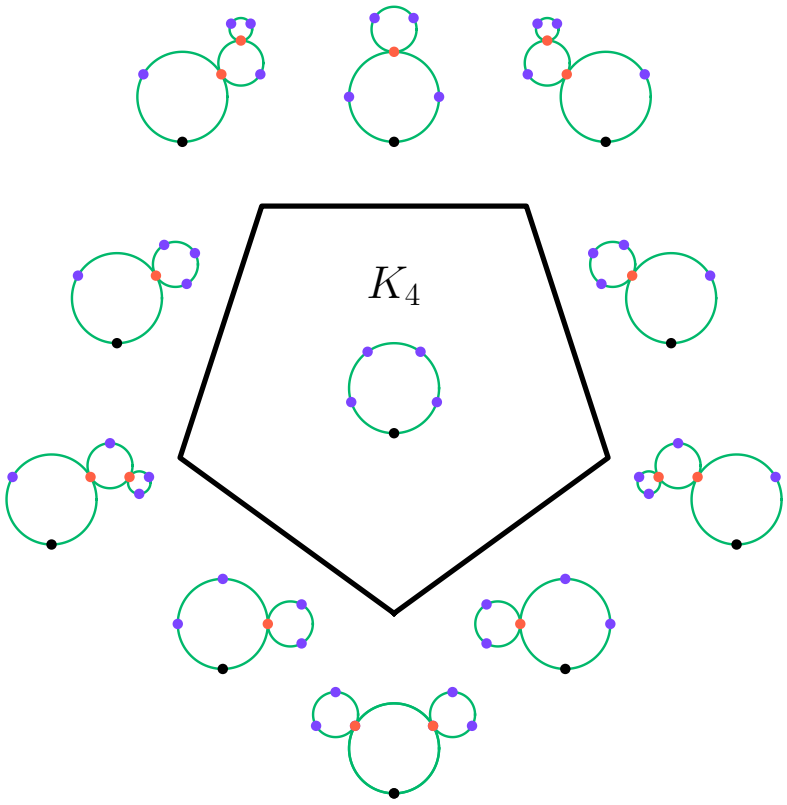


We have that (1) corresponds to a slit forming along L_3 that eventually reaches L_1 and thus cuts off the intersection with L_2 completely. Hence, the corresponding concatenation of A_∞-products is $\mu_2^F(x_0; \mu_2^F(x_3, x_2), x_1)$. Similarly, (2) and (3) correspond to $\mu_3^F(x_0; \mu_1^F(x_3), x_2, x_1)$ and $\mu_1^F(\mu_3^F(x_0; x_3, x_2, x_1))$ respectively.

By examining the correspondence between $\overline{\mathcal{R}}_{d+1}$ and K_d, we can identify these breakings with the breakings in the associahedron. For $\overline{\mathcal{R}}_4$, the corresponding associahedron K_3 is the interval



where we depicted the possible degenerations. Note that the left boundary corresponds to $\mu_2^F(x_0; x_3, \mu_2^F(x_2, x_1))$ and the right one to $\mu_2^F(x_0; \mu_2^F(x_3, x_2), x_1)$.[7] The other breakings, notably the semi-stable ones, are the result of strips breaking off in the interior of K_3. The case $d = 4$ is a little bit more interesting and we have the following stable breakings:



These diagrams are the key to the A_∞-relations of the wrapped Fukaya category; they provide the necessary cobordisms with boundaries the necessary components that make up the A_∞-relations. In general, the boundary $\overline{\mathcal{R}}_{d+1}$ strata that can be identified with compositions of $\mu_{d_1}^F$ and $\mu_{d_2}^F$ for $d_1, d_2 > 1$ and $d = d_1 + d_2$ correspond

[7] Recall that we number punctures clockwise.

to points of ∂K_d and strata that correspond to compositions of μ_1^F and μ_{d-1}^F are strip breakings in the interior. The next section will make this precise.

9.3 Floer Data for the A_∞-structure

In order to assign Floer data to punctured discs that are compatible with the breakings at the boundary $\overline{\mathcal{R}}_{d+1}$, and in order to generalise the transformation ψ^2 in the target of Map (8.2.10), we assign to each puncture ξ^i a weight w_i which is a positive real number. These weights play the following role: any equations that we shall consider on the punctured disc will pull back under the strip-like end ε^i to Equation (8.2.3) up to applying ψ^{w_i}. In the case of $\mathcal{R}_3$, with two incoming ends and one outgoing one, we have that the weights are equal to 1 for the inputs and to 2 for the outputs. Recall that ψ^z was defined to be the time-$\log(z)$ Liouville flow. If we consider an additional pullback by ψ^2 of all the data used to define the moduli space $\mathcal{R}_2(x_0; x_1, x_2)$ in Section 8.2.b), we obtain equations such that the weights are now equal to 2 on the inputs and 4 on the outputs. The space of solutions of the associated Cauchy-Riemann equation is a moduli space $\mathcal{R}_2(\psi^2(x_0); \psi^2(x_1), \psi^2(x_2))$. Counting its elements with orientations defines a modified version of Equation (8.2.10):

$$\mu_2^{\psi^2} : \mathrm{CW}_b^*(\psi^2(L_1), \psi^2(L_2)) \otimes \mathrm{CW}_b^*(\psi^2(L_0), \psi^2(L_1)) \to \mathrm{CW}_b^*(\psi^4(L_0), \psi^4(L_2)).$$

Note that this agrees with $\mathrm{CW}(\psi^{\frac{1}{4}}) \circ \mu_2^F \circ \mathrm{CW}(\psi^2)$. In the next section we will show that the following commuted up to homotopy:

$$\begin{array}{ccc}
\mathrm{CW}_b^*(\psi^2(L_1), \psi^2(L_3)) \otimes \mathrm{CW}_b^*(\psi^2(L_0), \psi^2(L_1)) & & \\
\uparrow & \searrow & \\
\mathrm{CW}_b^*(L_2, L_3) \otimes \mathrm{CW}_b^*(L_1, L_2) \otimes \mathrm{CW}_b^*(L_0, L_1) & & \mathrm{CW}_b^*(\psi^4(L_0), \psi^4(L_3)) \\
\downarrow & \nearrow & \downarrow \mathrm{CW}(\psi^{\frac{1}{4}}) \\
\mathrm{CW}_b^*(\psi^2(L_2), \psi^2(L_3)) \otimes \mathrm{CW}_b^*(\psi^2(L_0), \psi^2(L_2)) & & \\
 & & \mathrm{CW}_b^*(L_0, L_3).
\end{array} \tag{9.3.1}$$

The discussion of Section 9.2 then implies that such a homotopy should arise by considering the compactification of $\overline{\mathcal{R}}_3$. Any equation we impose on $\overline{\mathcal{R}}_3$ should restrict on the boundary strata to Equation (9.1.1). However, the Lagrangian boundary conditions used in the definitions of the equations on both components of the boundary do not agree. Nevertheless, they do agree up to applying ψ^2 to one of them; this is exactly incorporated in Diagram (9.3.1).

We could now play the same game as in Section 8.2.b) to define a map

$$\mathrm{CW}_b^*(L_2, L_3) \otimes \mathrm{CW}_b^*(L_1, L_2) \otimes \mathrm{CW}_b^*(L_0, L_1) \to \mathrm{CW}_b^*(L_0, L_3)$$

that provides the aforementioned homotopy. Since this is however just as involved as immediately generalising this to the n-th product, we will instead immediately

show it in the general case.

For this we first generalise transformations of the type ψ^2.

DEFINITION 9.3.1. *We say that a pair $(\rho_S^1, \alpha_S^1, H_S^1, I_S^1)$ and $(\rho_S^2, \alpha_S^2, H_S^2, I_S^2)$ of Floer data on a surface S are conformally equivalent if there exists a constant C such that ρ_S^2 and α_S^2 respectively agree with $C\rho_S^1$ and $C\alpha_S^1$, and*

$$I_S^2 = (\psi^C)^* I_S^1$$
$$H_S^1 = \frac{H_S^1 \circ \psi^C}{C^2}.$$

The proof of Lemma 8.2.10 then shows:

LEMMA 9.3.2 ([Abo09, Lem. 3.11.]). *Composition with ψ^C defines a bijective correspondence between solutions to Equation 9.1.1 for conformally equivalent Floer data.*

We can now make precise what was meant by stating that the Floer data on $\overline{\mathcal{R}}_d$ should be consistent with the Floer data coming from the breaking.

DEFINITION 9.3.3 ([Abo09, Def. 3.12]). *A universal and conformally consistent choice of Floer data D_μ for the A_∞-structure, is a choice of Floer data for every element of $\mathcal{R}_d$ and every integer $d \geq 2$, which varies smoothly over the interior of the moduli space, whose restriction to a boundary stratum is conformally equivalent to the product of Floer data coming from lower dimensional moduli spaces, and which near such a boundary stratum agrees to infinite order, in the coordinates (8.2.4), with the Floer data obtained by gluing.*

REMARK 9.3.4. Such a choice of Floer data D_μ fixes (up to constants) the choice of Floer data on $\partial\overline{\mathcal{R}}_d$ once Floer data on each irreducible component has been chosen. The discussion below Diagram (9.3.1) showed how in the case $d = 2$ a choice of Floer data for μ_2^F determined the Floer data on the boundary of $\overline{\mathcal{R}}_3$, namely by rescaling one of the discs in the concatenation of products by ψ^2. After introducing a perturbation of this Floer data that vanishes up to infinite order at the boundary, we can extend it from a neighbourhood of $\partial\overline{\mathcal{R}}_3$ to the remainder of $\mathcal{R}_3$. In such a manner, we can inductively construct a universal and consistent choice of Floer data D_μ using the fact that the boundary of the Stasheff associahedron, and thus also of $\overline{\mathcal{R}}_d$, can be covered by lower dimensional associahedra, i.e. there exist codimension 1 inclusions

$$\overline{\mathcal{R}}_{d_1} \times \overline{\mathcal{R}}_{d-d_1+1} \to \partial\overline{\mathcal{R}}_d,$$

together with the fact that the space of Floer data are contractible.

Furthermore, the contractibility of this space also implies that we may extend any Floer data chosen on a given surface to universal and consistent choices:

LEMMA 9.3.5 ([Abo09, Lem. 3.13.]). *The restriction map from the space of universal and conformally consistent Floer data to the space of Floer data for a fixed surface S is surjective.*

9.4 Higher Products

Subsequently, we will make true on our promise and define higher A_∞-products

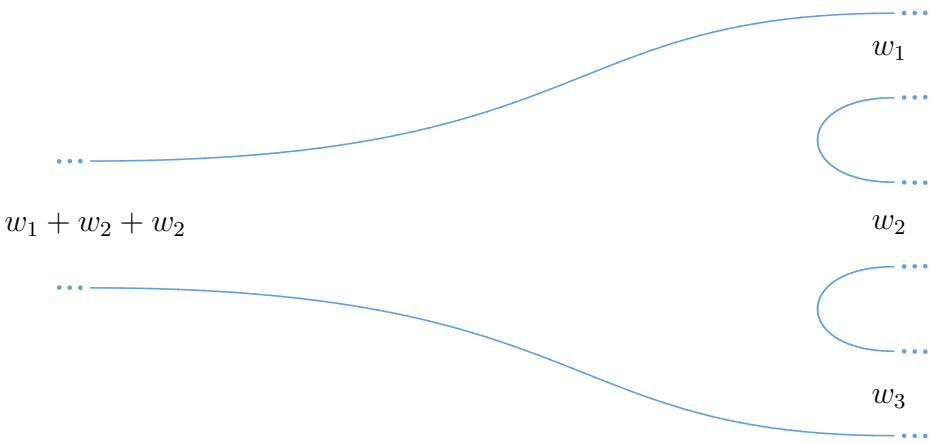
$$\mu_d^F : \mathrm{CW}_b^*(L_{d-1}, L_d) \otimes \cdots \otimes \mathrm{CW}_b^*(L_1, L_2) \otimes \mathrm{CW}_b^*(L_0, L_1) \to \mathrm{CW}_b^*(L_0, L_d) \quad (9.4.1)$$

coming from a weighted count of solutions to Equation (9.1.1). Let $\vec{x} := \{x_k \in \mathfrak{X}_H(L_{k-1}, L_k)\}$ be a sequence of Hamiltonian chords for $1 \leq k \leq d$ and $x_0 \in \mathfrak{X}_H(L_0, L_d)$ with $L_0, \ldots, L_d \in \mathrm{Obj}(\mathcal{W}_b(V))$. Define $\mathcal{R}_d(x_0; \vec{x})$ to be the space of solutions to Equation (9.1.1) whose source is an arbitrary element $S \in \mathcal{R}_d$ with marked points $(\xi^0, \ldots, \xi^d)$ such that

$$\lim_{s \to \pm\infty} u \circ \varepsilon^k(s, \cdot) = \psi^{w_{k,S}} \circ x_k$$

and with moving boundary conditions

$$u(z) \in \psi^{\rho_S(z)}(L_k) \quad \text{if } z \in \partial S \text{ lies between } \xi^k \text{ and } \xi^{k+1}.$$

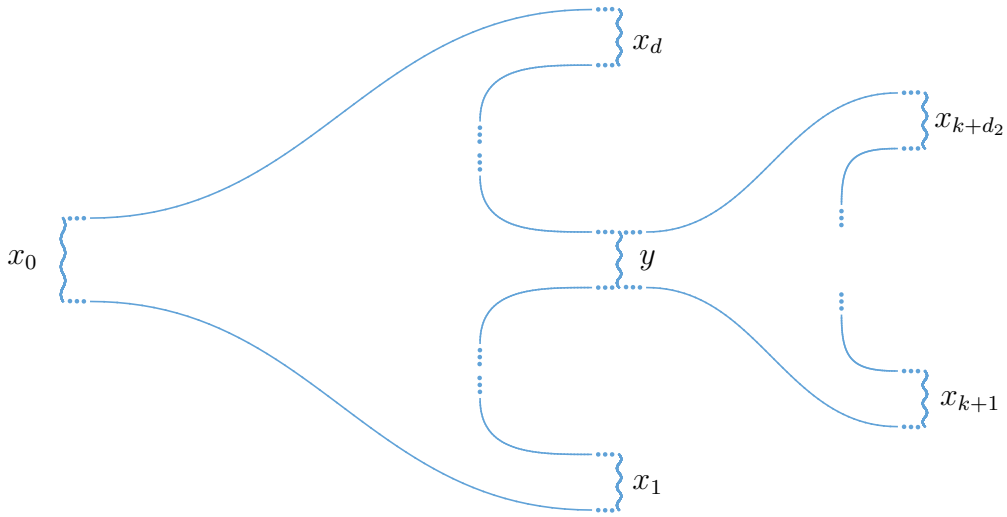


With the exception of semi-stable breakings, i.e. strip breakings, happening in the interior or $\mathcal{R}_d$, the (virtual) codimension 1 strata of the Gromov bordification of $\overline{\mathcal{R}}_d(x_0; \vec{x})$ lie over the codimension 1 strata of $\overline{\mathcal{R}}_d$. The consistency conditions on D_μ imply that whenever such a punctured disc breaks at the boundary, each component is a solution to Equation (9.1.1) for the Floer data included in D_μ up to applying ψ^C for some constant C that depends on the modulus in $\overline{\mathcal{R}}_d$. Since these broken discs can be identified with solutions of the original equation under composition with ψ^C, we are able to obtain a natural inclusion

$$\overline{\mathcal{R}}_{d_1}(x_0; \vec{x}^1) \times \overline{\mathcal{R}}_{d_2}(y; \vec{x}^2) \to \overline{\mathcal{R}}_d(x_0; \vec{x}) \quad (9.4.2)$$

for each integer $0 \leq k \leq d - d_2$ and Hamiltonian chords $y \in \mathfrak{X}_H(L_{k+1}, L_{l+d_2})$, where $\vec{x}^2 = (x_{k+1}, \ldots, x_{k+d_2})$ and $\vec{x}^1 = (x_1, \ldots, x_k, y, x_{k+d_2+1}, \ldots, x_d)$.

We can depict this by



We have:

LEMMA 9.4.1 ([Abo09, Lem. 3.14.]). *The moduli spaces $\overline{\mathcal{R}}_d(x_0;\vec{x})$ are compact, and are empty for all but finitely many x_0 once the inputs $\vec{x}$ are fixed. For a generic choice D_μ, they form manifolds of dimension*

$$|x_0| + d - 2 - \sum_{1\leq k\leq d} |x_k| \tag{9.4.3}$$

whose boundary is covered by the images of the natural inclusion (9.4.2).

The statement about the stratification of the compactification follows from the ideas presented at the end of Section 9.2, and the statement about the dimensions follows from a gluing argument presented in [Abo09, App. A.2]. The same gluing argument then guarantees for each element $u : S \to M$ the existence of an isomorphism

$$o_{\psi^{w_{d,S}}x_d} \otimes \cdots \otimes o_{\psi^{w_{1,S}}x_1} \to o_{\psi^{w_{0,S}}x_0}$$

whenever $|x_0| = 2 - d + \sum_{k=1}^{d} |x_k|$, i.e. when the moduli space is 0-dimensional. We will denote the map induced on orientation lines by μ_u and we define the d-th product

$$\mu_d^F : \mathrm{CW}_b^*(L_{d-1}, L_d) \otimes \cdots \otimes \mathrm{CW}_b^*(L_1, L_2) \otimes \mathrm{CW}_b^*(L_0, L_1) \to \mathrm{CW}_b^*(L_0, L_d),$$

by

$$\mu_d^F([x_d], \ldots, [x_1]) := \sum_{|x_0|=2-d+\sum\limits_{1\leq k\leq d}|x_k|} \; \sum_{u\in\mathcal{R}_d(x_0;\vec{x})} (-1)^{\blacklozenge} \mu_u([x_d], \ldots, [x_1]),$$

where $\blacklozenge = \sum_{k=1}^{d} k|x_k|$, and where we somewhat furtively oppressed the compositions with the isomorphism $\mathrm{CW}_b^*(\psi^{w_{k,S}})$ and $\mathrm{CW}_b^*(\psi^{w_{k,S}})^{-1}$.

As is customary by now, considering the moduli space of dimension 1, i.e. the case of $|x_0| = 2 - d + \sum |x_k| + 1$, we obtain the cobordism that is necessary to prove:

Proposition 9.4.2 ([Abo09, Prop. 3.15.]). *The collection of map μ_d^F defines an A_∞-structure on the category $\mathcal{W}_b(V)$. In particular, the sum*

$$\sum_{\substack{d_1+d_2=d+1\\0\leq k<d_1}} (-1)^{\blacktriangledown_k} \mu_{d_1}^F(x_d,\dots,x_{k+d_2+1},\mu_{d_2}^F(x_{k+d_2},\dots,x_{k+1}),x_k,\dots,x_1)=0$$

vanishes, where

$$\blacktriangledown_k := k+\sum_{1\leq j\leq k}|x_j|.$$

Proof (Sketch). Note that the discussion at the end of Section 9.2 guarantees that the boundaries of the 1-dimensional moduli spaces are given by products of the lower moduli spaces $\overline{\mathcal{R}}_{d'}$ with $d'<d$. The dimension formula (9.4.3) then provides that the punctured discs resulting from the breaking are rigid, and therefore correspond to the composition of operations $\mu_{d'}^F$ for $d'\geq 1$. The result then follows by taking signs into account. □

10 Construction of the A_∞-Functor

The goal of this section is to define an A_∞-functor between the wrapped Fukaya category and the twisted complex $\mathrm{Tw}(\mathcal{P}(M))$. We start by defining it at the level of objects.

10.1 Definition on Objects

We start with a set $\{q^i\}_{i=1}^m$ of intersection points between M and a Lagrangian L that is ordered by action. Note that since intersection points are also constant Hamiltonian chords, the action simplifies to the difference between f_M and f_L. Recall that $\mathcal{J}(V)$ was the space of almost complex structures on V that are compatible with the symplectic form on V, and such that

$$\lambda \circ J = dr$$

on the cylindrical end. Let $I_t \in \mathcal{J}(V)$ be the family of almost complex structures parametrised by $t \in [0,1]$, and consider the moduli spaces

$$\mathcal{H}(q^i, q^j)$$

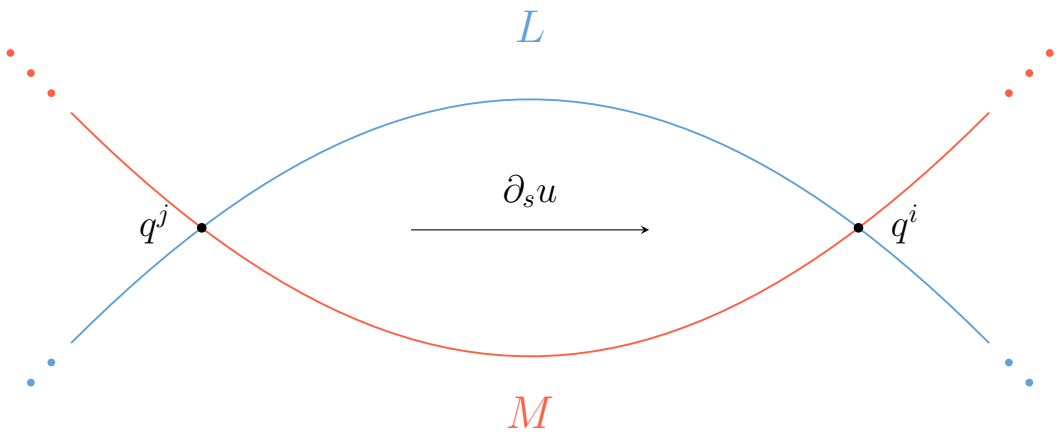
which are the quotients by the $\mathbb{R}$-action of the space of solutions to the Cauchy-Riemann equation:

$$Du(s,t) \circ j - I_t \circ Du(s,t) = 0 \tag{10.1.1}$$

for maps $u : \mathbb{R} \times [0,1] \to M$ satisfying the boundary conditions

$$\begin{aligned} u(\mathbb{R} \times \{1\}) &\subset L \\ u(\mathbb{R} \times \{0\}) &\subset M \\ \lim_{s \to -\infty} u(s,t) &= q^j \\ \lim_{s \to +\infty} u(s,t) &= q^i. \end{aligned}$$

This space is very similar to the moduli spaces before defined on the strip $\mathbb{R} \times [0,1]$, with the one key difference that we impose that the limit Hamiltonian chords is constant, i.e. intersection points.



As both Lagrangians M and L are graded by assumption, we may assign an index $|q|$ to each $q \in M \cap L$. A gluing argument presented in [Abo09, A.3.] shows the following:

PROPOSITION 10.1.1 ([Abo09, Prop. 2.5]). *For a generic family I_t, the Gromov compactification of $\mathcal{H}(q^i, q^j)$ is a topological manifold $\overline{\mathcal{H}}(q^i, q^j)$ of dimension $|q^i| - |q^j|$, possibly with boundary. The boundary is stratified into topological manifolds with the closure of codimension 1 strata given by the images of emdeddings*

$$\overline{\mathcal{H}}(q^i, q^{i_1}) \times \overline{\mathcal{H}}(q^{i_1}, q^j) \to \overline{\mathcal{H}}(q^i, q^j) \tag{10.1.2}$$

for all possible integers i_1 between i and j. For each pair $i_1 < i_2$ of such integers, we have a commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{H}}(q^i, q^{i_1}) \times \overline{\mathcal{H}}(q^{i_1}, q^{i_2}) \times \overline{\mathcal{H}}(q^{i_2}, q^j) & \longrightarrow & \overline{\mathcal{H}}(q^i, q^{i_1}) \times \overline{\mathcal{H}}(q^{i_1}, q^j) \\ \downarrow & & \downarrow \\ \overline{\mathcal{H}}(q^i, q^{i_2}) \times \overline{\mathcal{H}}(q^{i_2}, q^j) & \longrightarrow & \overline{\mathcal{H}}(q^i, q^j). \end{array} \tag{10.1.3}$$

$\square_{\text{ref.}}$

In order to relate $\overline{\mathcal{H}}(q^i, q^j)$ to $\Omega(q^i, q^j)$, we will need an evaluation map that takes pinched strips with end points q^i and q^j to paths from q^i to q^j. By elliptic regularity, the restriction of a solution u to Equation (10.1.1) to $\mathbb{R} \times \{0\}$ defines a smooth path from q^i to q^j; if we in addition reverse the direction in which we travel. In addition by [RS01, Thm. A] all of its derivatives decay exponentially at the ends in the C^∞-topology. Thus the arc-length R of $u|_{\mathbb{R}\times\{0\}}$ is well-defined and finite. Consequently we are able to define a continuous map $[0, R] \to M$ to each solution u. We have the following result:

LEMMA 10.1.2 ([Abo09, Lem. 2.7.]). *There exists a family of evaluation*

$$\overline{\mathcal{H}}(q^i, q^j) \to \Omega(q^i, q^j)$$

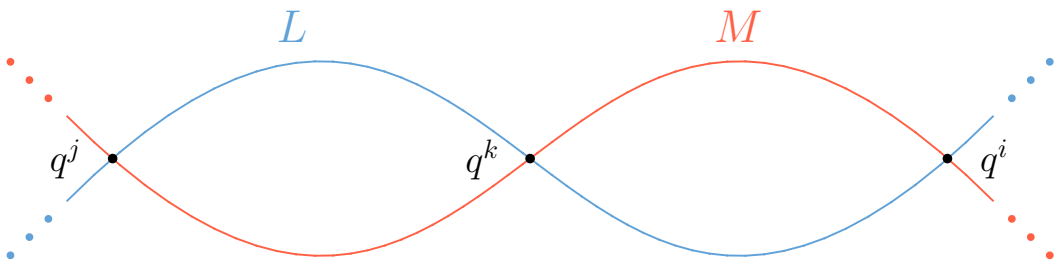
such that the following commutes:

$$\begin{array}{ccc} \overline{\mathcal{H}}(q^i, q^k) \times \overline{\mathcal{H}}(q^k, q^j) & \longrightarrow & \overline{\mathcal{H}}(q^i, q^j) \\ \downarrow & & \downarrow \\ \Omega(q^i, q^k) \times \Omega(q^k, q^j)) & \longrightarrow & \Omega(q^i, q^j), \end{array} \tag{10.1.4}$$

where the top arrow is the inclusion (10.1.2) and the bottom arrow is the concatenation of paths.

Proof. The construction of the evaluation map was done above. First, we have to show that the paths resulting from a breaking can be concatenated in a consistent manner, i.e. that the green arrow $\to$ is well-defined. Note that given a sequence $u_n \in \mathcal{H}(q^i, q^j)$, the phenomena of strips breaking can be explained by energy concentrating at one of the the respective ends $s \to \pm\infty$. There then exists a sequence

of real numbers $a_n \to \pm\infty$ such that the translated strips $u_n(s - a_n, t)$ converges to a non-constant limit strip, whose boundary we denote by q^k.



Estimate [MS12, Eq. 4.7.13] then ensures that near q^k, the norms of the derivatives of the family of holomorphic strips u_n decay exponentially. Writing (v_1, v_2) for the pair of strips into which u_n breaks, we thus have that the arc-lengths $R_1 = \operatorname{arc}(v_1|_{\mathbb{R}\times\{0\}})$ and $R_2 = \operatorname{arc}(v_2|_{\mathbb{R}\times\{0\}})$ are well-defined and finite. Thus we obtain the two continuous arc-length reparametrisations

$$[0, R_1] \to M \in \Omega(q^i, q^k)$$
$$[0, R_2] \to M \in \Omega(q^k, q^j)$$

of $v_1|_{\mathbb{R}\times\{0\}}$ and $v_2|_{\mathbb{R}\times\{0\}}$ respectively. The independence from the chosen sequence u_n is then immediate, as two different choices of sequences must converge to broken strips whose respective restriction to $\mathbb{R} \times \{0\}$ has the same arc-length. The commutativity is then also obvious. □

In order to prove some future results, we need the following statement about orientation lines:

Lemma 10.1.3 ([Abo09, Lem. A.3.]). *Choices of relative Pin structures on the Lagrangians L_k induce a canonical up to homotopy isomorphism*

$$\bigwedge^{\max}(\mathcal{R}_d(x_0; \vec{x})) \otimes o_{x_d} \otimes \cdots \otimes o_{x_1} \cong \bigwedge^{\max}(\mathcal{R}_d) \otimes o_{x_0}, \tag{10.1.5}$$

where we recall that $\bigwedge^{\max}$ denotes the top exterior power of the tangent bundle.

□ref.

With this, we can show:

Lemma 10.1.4 ([Abo09, Lem. 2.8.]). *The product orientation on $\overline{\mathcal{H}}(q^i, q^k) \times \overline{\mathcal{H}}(q^k, q^j)$ differs from its orientation as a boundary of $\overline{\mathcal{H}}(q^i, q^j)$ by a sign of parity $|q^i| + |q^j|$.*

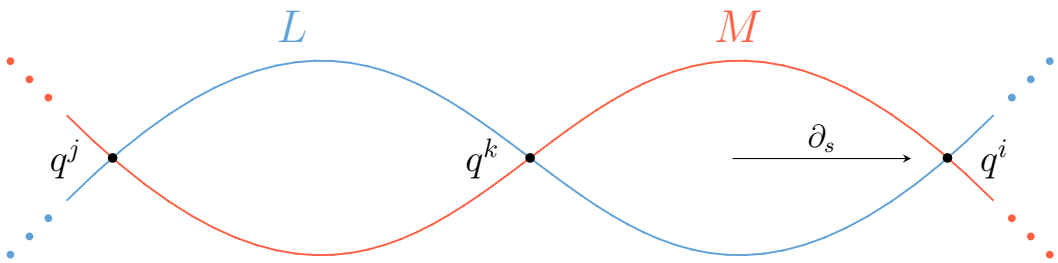
Proof. Note that $\overline{\mathcal{H}}(q^i, q^j)$ can be seen as a special case of $\mathcal{R}_1(x_0, x_1)$ with constant limit chords. Thus (10.1.5) transforms into

$$\begin{aligned} &\bigwedge^{\max}(\overline{\mathcal{H}}(q^i, q^j)) \otimes o_{q^j} \cong |\mathbb{R} \cdot \partial_s| \otimes o_{q^i} \\ \implies &\bigwedge^{\max}(\overline{\mathcal{H}}(q^i, q^j)) \cong |\mathbb{R} \cdot \partial_s| \otimes o_{q^i} \otimes o_{q^j}^{-1}. \end{aligned} \tag{10.1.6}$$

Taking the tensor product of these isomorphisms for the pairs (q^i, q^k) and (q^k, q^j), one obtains

$$\bigwedge^{\max}(\overline{\mathcal{H}}(q^i, q^k)) \otimes \bigwedge^{\max}(\overline{\mathcal{H}}(q^k, q^j)) \cong |\mathbb{R} \cdot \partial_s| \otimes o_{q^i} \otimes o_{q^k}^{-1} \otimes |\mathbb{R} \cdot \partial_s| \otimes o_{q^k} \otimes o_{q^j}^{-1}. \tag{10.1.7}$$

The key observation is now that the vector field coming from $|\mathbb{R} \cdot \partial_s|$ corresponds to an outward pointing vector, see [Sei08, Sec.(12f)].



We can thus identify it with the normal vector ν, reorder the terms of (10.1.7), and cancel the copy of $o_{q^k}^{-1} \otimes o_{q^k}$, we obtain:

$$\bigwedge^{\max}(\overline{\mathcal{H}}(q^i, q^k)) \otimes \bigwedge^{\max}(\overline{\mathcal{H}}(q^k, q^j)) \cong \nu \otimes |\mathbb{R} \cdot \partial_s| \otimes o_{q^i} \otimes o_{q^j}^{-1}$$

with a Koszul sign of

$$(-1)^{|q^k|+|q^i|},$$

resulting from bringing $|\mathbb{R} \cdot \partial_s$ forward. □

By this result, we are able to choose fundamental chains for $[\overline{\mathcal{H}}(q^i, q^j)]$ inductively in the cubical chain complex $C_*(\overline{\mathcal{H}}(q^i, q^k))$. We start by picking representatives of the fundamental cycles of all components of $\bigcup \overline{\mathcal{H}}(q^i, q^j)$ which are manifolds without boundary that are compatible with the orientations determined by our choice of brane data (recall that this was a choice of grading on each Lagrangian, together with a relative Pin structure), and the isomorphism of Equation (10.1.6).

We can now proceed by induction, assuming that a class satisfying

$$\partial[\overline{\mathcal{H}}(q^i, q^k)] = \sum_k (-1)^{|q^i|+|q^k|}[\overline{\mathcal{H}}(q^i, q^k)] \times [\overline{\mathcal{H}}(q^k, q^j)] \tag{10.1.8}$$

was chosen for all moduli spaces of dimension smaller than some integer m, where the sign $(-1)^{|q^i|+|q^k|}$ accounts for the sign discrepancy explained in Lemma 10.1.4. In addition, we see $[\overline{\mathcal{H}}(q^i,q^k)]\times[\overline{\mathcal{H}}(q^k,q^j)]$ as an element of $C_*(\overline{\mathcal{H}}(q^i,q^j))$ by Inclusion (10.1.2).

For the inclusion step, assume that we are given $\overline{\mathcal{H}}(q^i,q^j)$ such that the dimension of this moduli space is equal to $m+1$. The associativity of the cross product (10.1.3) and the commutativity of diagram (10.1.4) imply that

$$\partial\Big(\sum_k (-1)^{|q^i|+|q^k|}[\overline{\mathcal{H}}(q^i,q^k)]\times[\overline{\mathcal{H}}(q^k,q^j)]\Big)=0.$$

As this is a closed class of codimension 1 supported on the boundary of $[\overline{\mathcal{H}}(q^i,q^j)]$, we may choose a bounding class $[\overline{\mathcal{H}}(q^i,q^j)]$

LEMMA 10.1.5 ([Abo09, Lem. 2.9.]). *Each Lagrangian $L\in\mathrm{Obj}(\mathcal{W}_b(V))$ determines a twisted complex $\mathcal{F}(L)$ in $\mathrm{Tw}(\mathcal{P}(M))$ given by*

$$\Big(\bigoplus_{q^i\in M\cap L} q^i[-|q^i|],\sum_{q^i,q^j}(-1)^{|q^i|(|q^j|+1)}[\overline{\mathcal{H}}(q^i,q^j)]\Big).$$

Proof. In order to show the claim, we have to show that Equation (8.1.5) holds, which in this case takes on the form:

$$(-1)^{|q^i|(|q^j|+1)}\partial[\overline{\mathcal{H}}(q^i,q^j)]+\sum_k(-1)^*[\overline{\mathcal{H}}(q^i,q^k)]\times[\overline{\mathcal{H}}(q^k,q^j)]=0,$$

where the sign $*$ consists of the sum of the following signs:

$$\begin{aligned}|q^i|(|q^k|+1)+|q^k|(|q^j|+1)&\text{ from the definition of the twisted complex,}\\ (\dim(\overline{\mathcal{H}}(q^k,q^j))+1)(|q^k|-|q^i|)&\text{ from Equation (8.1.4),}\\ \dim(\overline{\mathcal{H}}(q^i,q^k))&\text{ from the definition of the differential (8.1.1).}\end{aligned}$$

Using that $\dim(\overline{\mathcal{H}}(q^k,q^j))=|q^k|-|q^j|$, we see that the sum of $*$ with $(|q^i|(|q^j|+1)$ is equal to $1+|q^i|+|q^k|$, which is exactly the opposite of the sign of (10.1.8). □

10.2 Definition on Morphisms

In this section, we will extend the map $\mathcal{F}$ to an A_∞-functor. We start by defining a chain map

$$\mathcal{F}^1:\mathrm{CW}_b^*(L_0,L_1)\to\mathrm{Hom}_*(\mathcal{F}(L_0),\mathcal{F}(L_1)). \tag{10.2.1}$$

10.2.a) Moduli Spaces of Half-Strips

In order to construct the above chain map, we need to define a new moduli space. On the half-strip $Z^+=[0,+\infty)\times[0,1]$, define the punctures $\xi^0=(0,1)$ and $\xi^1=(0,0)$ and consider the surface $T:=Z^+\setminus\{\xi^0,\xi^{-1}\}$. Alternatively, we can think of T as a 3 punctured disc, and we will denote the puncture corresponding to the infinite part of T by ξ^1. We shall consider a family of Hamiltonians

$$H_T:T\to\mathscr{H}(V)$$

parametrised by points of T, that agree near ξ^1 with H and whose restriction to the interval between ξ^0 and ξ^1 takes values in $\mathscr{H}_M(V)$, i.e. maps that vanish on M.

We shall impose the following Cauchy-Riemann equation:

$$(Du - X_t \otimes d\tau)^{0,1} = 0 \tag{10.2.2}$$

where X_T is the Hamiltonian flow of H_T, and $u : T \to V$. To specify the choice of a family of almost complex structures with respect to which the Cauchy-Riemann equation is defined, let us fix a positive strip-like end ε^0 near ξ^0 and a negative end ε^{-1} near ξ^{-1}. We can then choose a family of almost complex structures

$$I_t : T \to \mathcal{J}(V)$$

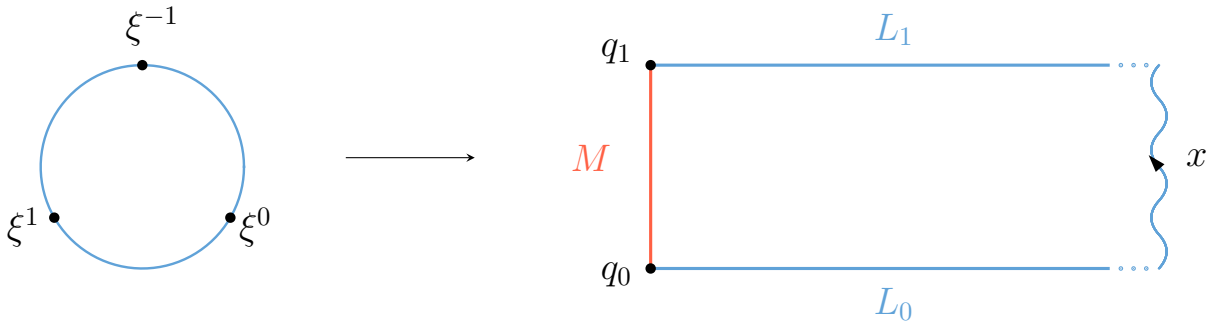
which agree with I_t near ξ^1, and whose pullback under ε^0 and ε^1 also agrees with I_t. We then have that the Cauchy-Riemann Equation (10.2.2) agrees with Equation (8.2.3) near ξ^1 and that its pullback under the strip-like ends ε^0 and ε^1 agrees with Equation (10.1.1).

Given a chord $x \in \mathfrak{X}_H(L_0, L_1)$ and intersection points q_0 between L_0 and M and q_1 between L_1 and M, it then makes sense to define $\mathcal{H}(q_0, x, q_1)$ to be the moduli space of solutions u to Equation (10.2.2) with boundary conditions

$$\begin{cases} u(z) \in L_0 & \text{if } z \in \partial T \text{ lies on the segment joining } \xi^0 \text{ and } \xi^1, \\ u(z) \in L_1 & \text{if } z \in \partial T \text{ lies on the segment joining } \xi^{-1} \text{ and } \xi^1, \\ u(z) \in M & \text{if } z \in \partial T \text{ lies on the outgoing segment,} \end{cases}$$

and asymptotic conditions

$$\begin{cases} \lim_{s \to +\infty} u(s, \cdot) = x(\cdot) \\ \lim_{s \to -\infty} u \circ \varepsilon^1(s, t) = q_1 \\ \lim_{s \to +\infty} u \circ \varepsilon^0(s, t) = q_0 \end{cases}$$



By transversality results we have:

Lemma 10.2.1 ([Abo09, Lem. 4.2]). *For generic choices of Hamiltonians H_T and almost complex structures J_T, the moduli space $\mathcal{H}(q_0, x, q_1)$ is a smooth manifold of dimension*

$$|q_0| - |q_1| - |x|.$$

□ref.

Since T can be identified with the 3-punctured disc, there are only 3 types of degenerations that are possible, namely strip breakings at punctures.

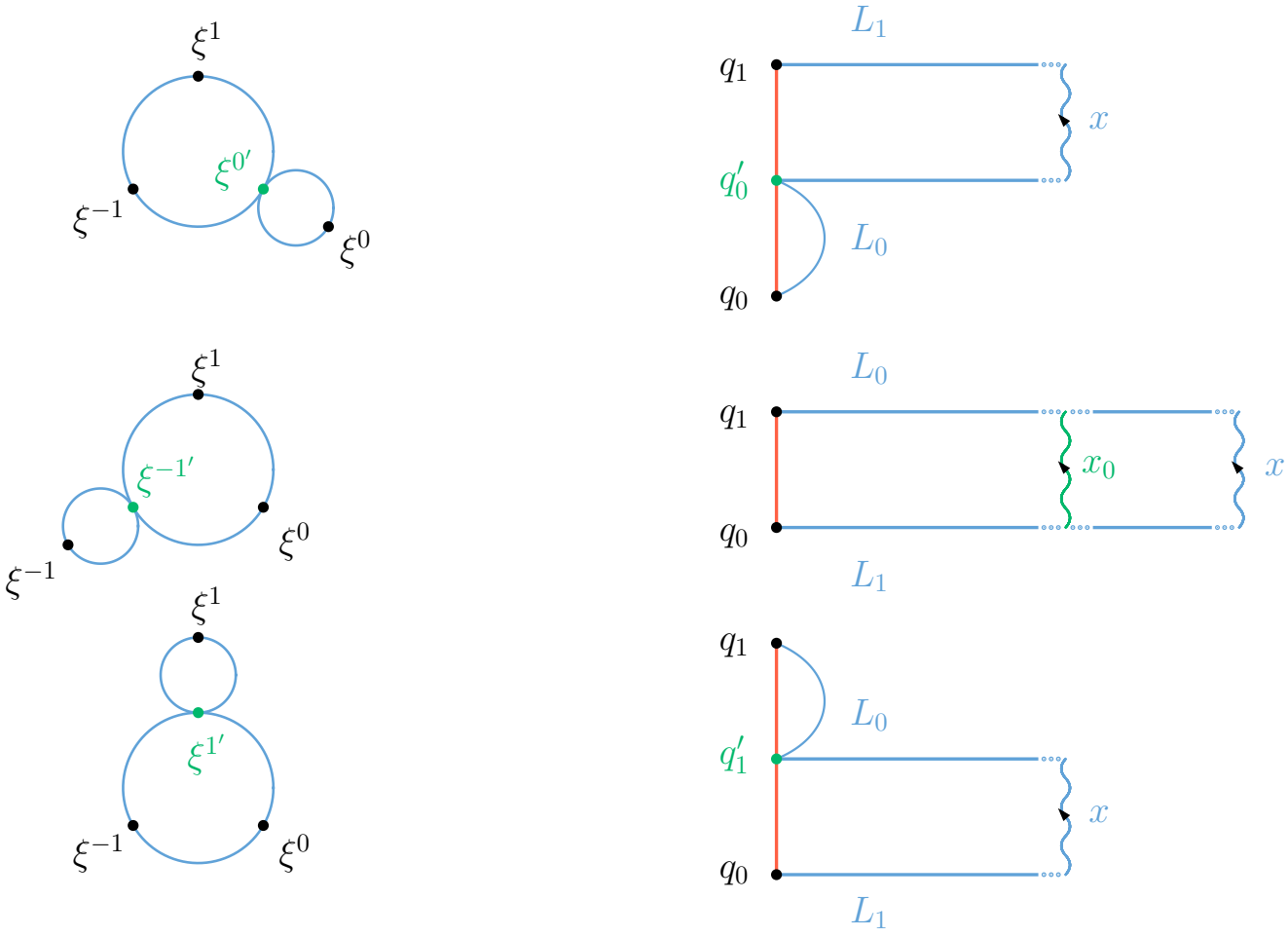


Figure II.3: Boundary strata of $\overline{\mathcal{H}}(q_0, x, q_1)$.

The virtual codimension 1 strata are then given by

$$\bigsqcup_{q_0' \in L_0 \cap M} \mathcal{H}(q_0, q_0') \times \mathcal{H}(q_{0'}, x, q_1), \tag{10.2.3}$$

$$\bigsqcup_{q_1' \in L_1 \cap M} \mathcal{H}(q_0, x, q_1') \times \mathcal{H}(q_1', q_1), \tag{10.2.4}$$

$$\bigsqcup_{x_0 \in \mathfrak{X}_H(L_0, L_1)} \mathcal{H}(q_0, x_0, q_1) \times \mathcal{R}(x_0, x). \tag{10.2.5}$$

We then have:

Lemma 10.2.2 ([Abo09, Lem. 4.3.]). *The Gromov compactification $\overline{\mathcal{H}}(q_0, x, q_1)$ is a compact manifold whose boundary is covered by the image of the inclusions of the*

strata above.

$\square_{\text{ref.}}$

The following analogue of Lemma 10.1.4 holds.

LEMMA 10.2.3 ([Abo09, Lem. 4.4.]). *The difference between the orientation induced as a boundary of $\overline{\mathcal{H}}(q_0, x, q_1)$ and the product orientation is given by signs whose parity is*

$$\begin{aligned} |x_0| + |q_0| \quad & \text{for the strata (10.2.5) if } \mathcal{R}(x_0, x) \text{ is rigid,} \\ 1 + |q_0| + |x| + |q_1'| \quad & \text{for the strata (10.2.4) ,} \\ 0 \quad & \text{for the strata (10.2.3) .} \end{aligned}$$

$\square_{\text{ref.}}$

Similarly to before, we now want to define some fundamental cycles that enable us to define $\mathcal{F}^1$. Consider the evalutaion maps

$$\text{ev} : \overline{\mathcal{H}}(q_0, x, q_1) \to \Omega(q_0, q_1)$$

which takes every solution u of (10.2.2) to the path along M between q_0 and q_1 obtained by restricting u to the outgoing segment of ∂T. As before, we then reparametrise this segment in M by its arc-length to obtain a path in $\Omega(q_0, q_1)$.

A similar argument as the one in the proof of Lemma 10.1.2 for the boundary strata (10.2.5) leads us to the commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{H}}(q_0, x_0, q_1) \times \mathcal{R}(x_0, x) & \longrightarrow & \overline{\mathcal{H}}(q_0, x, q_1) \\ \downarrow & & \downarrow \\ \overline{\mathcal{H}}(q_0, x_0, q_1) & \longrightarrow & \Omega(q_0, q_1), \end{array}$$

in which the left vertical arrow is the projection onto the first factor, and the top arrow is the inclusion into the boundary stratum (10.2.5). Note that the second factor $\mathcal{R}(x_0, x)$ does not not contribute to the evaluation map as one can make sure by looking at the middle of Figure II.3. Analysing the strata (10.2.4) and (10.2.3) yields

$$\begin{array}{ccccc} \mathcal{H}(q_0, x_0, q_1') \times \mathcal{H}(q_1', q_1) & \longrightarrow & \overline{\mathcal{H}}(q_0, x, q_1) & \longleftarrow & \mathcal{H}(q_0, q_0') \times \mathcal{H}(q_0', x, q_1) \\ \downarrow{\scriptstyle \text{ev} \times \text{ev}} & & \downarrow{\scriptstyle \text{ev}} & & \downarrow{\scriptstyle \text{ev} \times \text{ev}} \\ \Omega(q_0, q_1') \times \Omega(q_1', q_1) & \longrightarrow & \Omega(q_0, q_1) & \longleftarrow & \Omega(q_0, q_0') \times \Omega(q_0', q_1), \end{array}$$

where again the top arrows are inclusions into the boundary of the compactification, and the bottom arrows are concatenations of paths.

We now want to construct fundamental chains for $\mathcal{H}(q_0, x, q_1)$. We start with the previously chosen chains for the moduli spaces $\overline{\mathcal{H}}(q_i, q_j)$ just before Lemma 10.1.5. We can then repeat this procedure to choose fundamental chains for the moduli spaces $\overline{\mathcal{R}}(x_0, x)$. Next, we choose fundamental cycles for those moduli spaces

$\overline{\mathcal{H}}(q_0, x, q_1)$ which do *not* have boundary compatibility with orientations, which we will then subsequently rectify.

Lemma 10.2.3 tells us what the discrepancy of the product orientation of (10.2.3)-(10.2.5) is compared to the induced orientations as the boundary strata of $\overline{\mathcal{H}}(q_0, x, q_1)$. We can thus construct a chain in $C_*(\overline{\mathcal{H}}(q_0, x, q_1)$ that is boundary compatible with orientations:

$$\sum_{x_0 \in \mathfrak{X}_H(L_0,L_1)} \pm [\mathcal{H}(q_0, x_0, q_1)] + [\mathcal{R}(x_0, x)] + \sum_{q_0' \in L_0 \cap M} [\mathcal{H}(q_0, q_0')] \times [\mathcal{H}(q_0', x, q_1)] + \sum_{q_1' \in L_1 \cap M} (-1)^{1+|q_0|+|x|+|q_1'|} [\mathcal{H}(q_0, x, q_1')] \times [\mathcal{H}(q_1', q_1)], \tag{10.2.6}$$

where the sign in the first summation is $|x_0| + |q_0|$ if $\mathcal{R}(x_0, x)$ is rigid, and does not enter in the following constructions otherwise.

Lemma 10.2.4 then implies that (10.2.6) is a cycle, and we set the fundamental chain $[\mathcal{H}(q_0, x, q_1)]$ to be any chain whose boundary is Equation (10.2.6).

LEMMA 10.2.4 ([Abo09, Lem. 4.5.]). *There exist fundamental chains $[\overline{\mathcal{H}}(q_0, x, q_1)] \in C_*(\overline{\mathcal{H}}(q_0, x, q_1)$ such that the assignment*

$$\mathcal{F}^1([x]) = \bigoplus_{q_0, q_1} (-1)^{|x|+(|q_0|+1)(|x|+|q_1|)} \operatorname{ev}_*([\overline{\mathcal{H}}(q_0, x, q_1)])$$

defines the chain map described in Equation (10.2.1).

Proof. We prove this lemma by considering the strata (10.2.3)-(10.2.5) one by one. The easiest case is the one of (10.2.5), where $\mathcal{R}(x_0, x)$ is not rigid. Since the fundamental chains of this stratum are obtained by taking the products of the fundamental chains on each factor, its image in $C_*(\Omega(q_0, q_1))$ is degenerate; and hence vanishes. Thus such strata do not contribute to $\partial \operatorname{ev}_*([\overline{\mathcal{H}}(q_0, x, q_1)])$.

Consulting the definition of the differential of the twisted complex (8.1.7), in addition to the definition of $\mathcal{F}$ on objects in Lemma 10.1.5, we see that showing that $\mathcal{F}^1$ is a chain map is equivalent to showing

$$\begin{aligned}\mathcal{F}^1(\mu_1^F[x]) &\stackrel{!}{=} \mu_1^{\mathrm{Tw}(\mathcal{P})}(\mathcal{F}^1[x]) \\ &:= \mu_1^P(\mathcal{F}^1[x]) + \mu_2^P\Big(\mathcal{F}^1[x], \sum_{q_0, q_0'} [\overline{\mathcal{H}}(q_0, q_0')]\Big) + \mu_2^P\Big(\sum_{q_1, q_1'} [\overline{\mathcal{H}}(q_1', q_1)], \mathcal{F}^1[x]\Big).\end{aligned}$$

Recall that we set $\mu_1^P \sigma = \partial \sigma$ and thus we have by defintion that

$$\partial \operatorname{ev}_*([\overline{\mathcal{H}}(q_0, x, q_1)]) = \mu_1^P(\mathcal{F}^1[x]).$$

Analysing the other strata will yield the rest. For rigid moduli spaces $\mathcal{R}(x_0, x)$, the stratum $\mathcal{H}(q_0, x_0, q_1) \times \mathcal{R}(x_0, x)$ corresponds to $\mathcal{F}^1(\mu_1^F[x])$; see the middle of Figure II.3. Considering the remaining strata (10.2.4) and (10.2.3) together with the other two breakings of Figure II.3, we obtain that (10.2.4) corresponds to $\mu_2^P\Big(\sum_{q_1, q_1'} [\overline{\mathcal{H}}(q_1', q_1)], \mathcal{F}^1[x]\Big)$ and (10.2.3) corresponds to $\mu_2^P\Big(\mathcal{F}^1[x], \sum_{q_0, q_0'} [\overline{\mathcal{H}}(q_0, q_0')]\Big)$. □

10.2.b) Moduli spaces of Half-Discs

We will now generalise $\mathcal{F}^1$ to a functor

$$\mathcal{F}^d : \mathrm{CW}_b^*(L_{d-1}, L_d) \otimes \cdots \otimes \mathrm{CW}_b^*(L_1, L_2) \otimes \mathrm{CW}_b^*(L_0, L_1) \to \mathrm{Hom}_*(\mathcal{F}(L_0), \mathcal{F}(L_d)) \tag{10.2.7}$$

for $d > 1$. These maps will then together form our final A_∞-functor. As it turns out, the proofs will be of very similar nature to the ones of the previous section, and we will thus skip most of the proofs. However, setting up the correct moduli space presents some technical difficulties similar to the ones presented in Section 9.4 and thus we will establish it in full detail.

Let us denote by $\mathcal{H}_d$ the moduli space of holomorphic discs with $d+2$ marked points, of which d successive ones are marked as incoming and the remaining two as outgoing points. By looking at the structure of the Stasheff associahedron, we can identify $\mathcal{H}_0$ with a point, corresponding to the moduli space of strips. We then fix an orientation of $\mathcal{H}_d$ using the conventions for the Stasheff associahedron used throughout [Sei08], and the isomorphism

$$\mathcal{H}_d \cong \mathcal{R}_{d+1} \tag{10.2.8}$$

taking the incoming marked points on the source to the first d incoming marked points on the target. We will now give a more geometric interpretation of the Deligne-Mumford compactification of $\mathcal{H}_d$ into $\overline{\mathcal{H}}_d$.

If a breaking occurs *away* from the outgoing segment, we have that the boundary stratum corresponding to this breakings is covered by products of the form

$$\overline{\mathcal{H}}_{d_1} \times \overline{\mathcal{R}}_{d_2} \to \overline{\mathcal{H}}_d$$

with $d_1 + d_2 = d + 1$.

If the breaking takes place on the outgoing segment, the strata is covered by products

$$\overline{\mathcal{H}}_{d_1} \times \overline{\mathcal{H}}_{d_2} \to \overline{\mathcal{H}}_d$$

with $d = d_1 + d_2$. These two types of breakings can best be understood pictorially

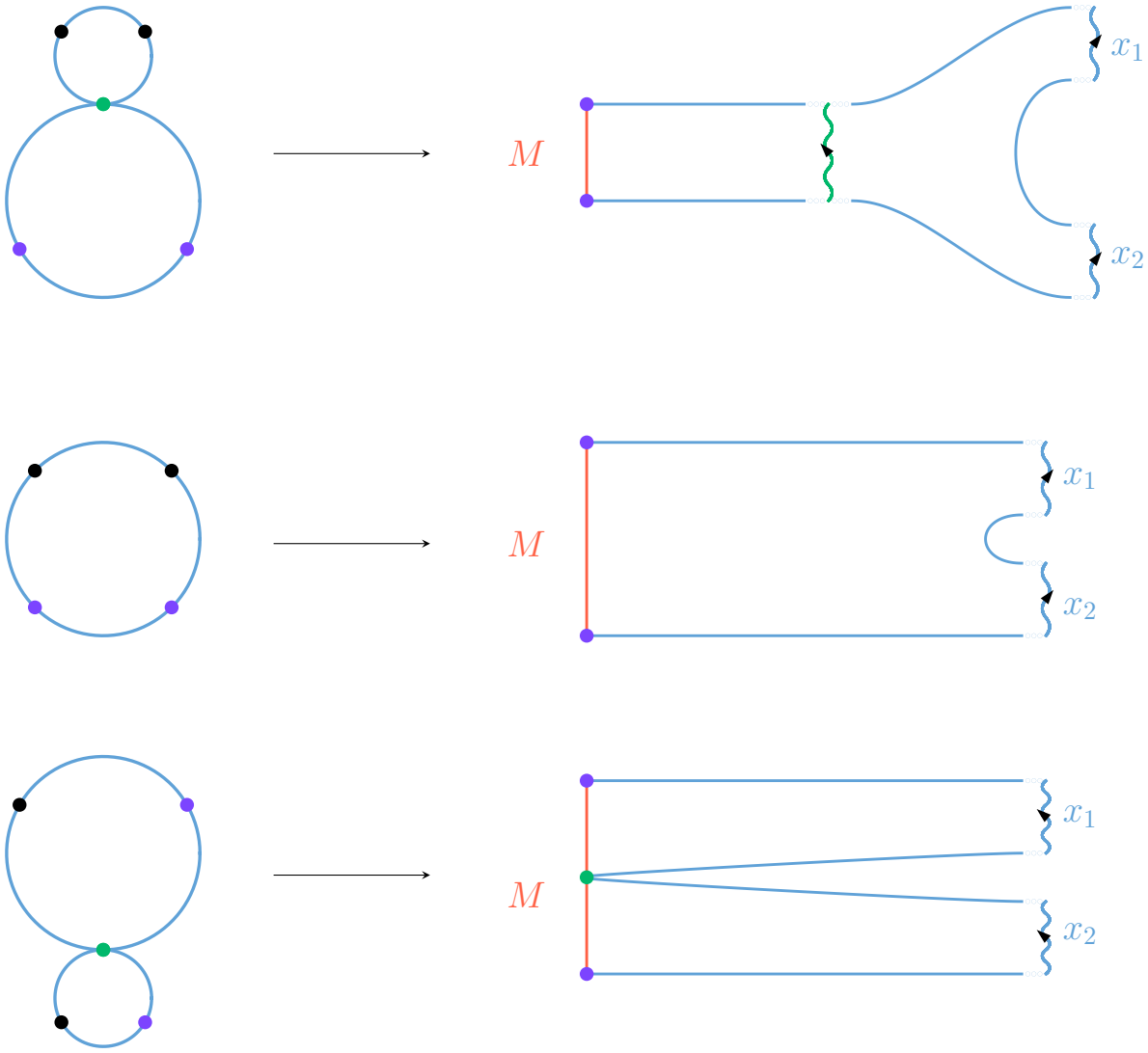


Figure II.4: Different breakings for $\mathcal{H}_d$. The purple points mark the outgoing points.

Via Isomorphism (10.2.8), a universal choice of strip-like ends on $\mathcal{R}_d$ induces one on $\mathcal{H}_{d-1}$.

Definition 10.2.5. *A* ***Floer datum*** D_T *on a stable disc* $T \in \overline{\mathcal{H}}_d$ *consists of the following choices on each component:*

i) Weights: a positive real number $w_{k,T}$ *associated to each end of* T *which is assumed to be 1 for* $k = -1, 0$.

ii) Time shifting maps: a map $\rho_T : \partial T \to [1, +\infty)$ *which agrees with* $w_{k,T}$ *near* ξ^k.

iii) Hamiltonian perturbations: a map $H_T : T \to \mathscr{H}(V)$ on each surface such that the restriction of H_T to a neighbourhood of the outgoing boundary segment takes value in $\mathscr{H}_M(V)$, and whose value near the ξ^k for $1 \leq k \leq d$ is

$$\frac{H \circ \psi^{w_{k,T}}}{w_{k,T}^2}.$$

iv) Basic 1-form: a closed 1-form α_T whose restriction to the complement of the outgoing segment in ∂T and to a neighbourhood of ξ^0 and ξ^{-1} vanishes, and whose pullback under ε^k for $1 \leq k \leq d$ agrees with $w_{k,T} d\tau$.

v) Almost complex structure: a map $I_T : T \to \mathcal{J}(V)$ whose pullback under ε^k agrees with $(\psi^{w_{k,T}})^* I_t$.

Note that the closedness of α_T and the fact that it vanishes near the outgoing marked points implies that it does not vanish on the outgoing boundary segment. However, since the outgoing segment is mapped to a compact Lagrangian, this does not cause problems with compactness.

The imposed Cauchy-Riemann equation does not change, it is still

$$(Du - X_T \otimes \alpha_T)^{0,1} = 0, \tag{10.2.9}$$

where the $(0,1)$-part is taken with respect to the almost compact structure I_t. Moreover, we still have that the pullback of Equation (10.2.9) under the end ε^k agrees with Equation (10.1.1) for $k = -1, 0$, and is otherwise equal to

$$(Du \circ \varepsilon^k - X_{\frac{H \circ \psi^{w_{k,T}}}{w_{k,T}}} \otimes d\tau)^{0,1} = 0.$$

Analogously to Definition 9.3.3, we want our choices of Floer data for the different degrees to be consistent with breakings. We can now choose the Floer data inductively by taking into account that we are dealing with moving Lagrangian boundary conditions.

Definition 10.2.6 ([Abo09, Def. 4.9.]). *A **universal and conformally** consistent choice of Floer data for the homomorphism $\mathcal{F}$, is a choice $D_{\mathcal{F}}$ of such Floer data for every integer $d \geq 1$, and every (representative) of an element of $\mathcal{H}_d$ which varies smoothly over this compactified moduli space, whose restriction to a boundary stratum is conformally equivalent to the product of Floer data coming from either D_μ or a lower dimensional moduli space $\mathcal{H}_d$, and which near such a boundary stratum agrees to infinite order with the Floer data obtained by gluing.*

Given a sequence of Hamiltonian chords $\vec{x} := \{x_k \in \mathfrak{X}_H(L_{k-1}, L_k)\}$ if $1 \leq k \leq d$, and a pair of intersection points $q_0 \in L_0 \cap M$ and $q_d \in L_d \cap M$, we set $\mathcal{H}(q_0, \vec{x}, q_d)$ to be the moduli space of solutions to Equation (10.2.9) whose source is an arbitrary element $T \in \mathcal{H}_d$ with boundary conditions

$$\begin{cases} u(z) \in \psi^{\rho(z)}(L_0) & \text{if } z \in \partial T \text{ lies on the segment between } \xi^0 \text{ and } \xi^1, \\ u(z) \in \psi^{\rho(z)}(L_k) & \text{if } z \in \partial T \text{ lies on the segment between } \xi^k \text{ and } \xi^{k+1}, \\ u(z) \in \psi^{\rho(z)}(L_d) & \text{if } z \in \partial T \text{ lies on the segment between } \xi^{-1} \text{ and } \xi^d, \\ u(z) \in M & \text{if } z \in \partial T \text{ lies on the outgoing segment;} \end{cases}$$

together with asymptotic conditions

$$\begin{cases} \lim_{s\to+\infty} u(\varepsilon^k(s,t)) = x_k(t) \ \text{ if } 1 \leq k \leq d \\ \lim_{s\to+\infty} u(\varepsilon^0(s,t)) = q_0 \\ \lim_{s\to-\infty} u(\varepsilon^0(s,t)) = q_d. \end{cases}$$

We have:

LEMMA 10.2.7 ([Abo09, Lem. 4.10.]). *For generic choices of universal Floer data, $\mathcal{H}(q_0, \vec{x}, q_d)$ is a smooth manifold of dimension*

$$d - 1 + |q_0| - |q_d| - \sum_{k=1}^{d} |x_k|.$$

The Gromov compactification of the abstract moduli space of half-discs extends to a Gromov compactification of the moduli spaces $\mathcal{H}(q_0, \vec{x}, q_d)$. There are again two types of breakings that can occur. One, if the sequence of holomorphic half-discs breaks away from the outgoing points. Then there exist two integers $d_1 + d_2 = d + 1$ together with a partition of the sequence $\vec{x}$ of Hamiltonian chords into $\vec{x}_1 = (x_1, \ldots, x_k, y, x_{k+d_2+1}, \ldots, x_d)$ with $y \in \mathfrak{X}_H(L_k, L_{k+d_2})$ and $\vec{x}_2 = (x_{k+1}, \ldots, x_{k+d_2})$ such that the corresponding boundary strata of $\overline{\mathcal{H}}(q_0, \vec{x}, q_d)$ is covered by

$$\overline{\mathcal{H}}(q_0, \vec{x}_1, q_d) \times \overline{\mathcal{R}}(y, \vec{x}_2) \to \overline{\mathcal{H}}(q_0, \vec{x}, q_d). \tag{10.2.10}$$

Similarly breaking between the outgoing points induce splittings

$$\overline{\mathcal{H}}(q_0, \vec{x}_1, q'_{d_1}) \times \overline{\mathcal{H}}(q'_{d_1}, \vec{x}_2, q_d) \to \overline{\mathcal{H}}(q_0, \vec{x}, q_d), \tag{10.2.11}$$

where $\vec{x}_1 = \{x_1, \ldots, x_{d_1}\}$, $\vec{x}_2 = \{x_{d_1+1}, \ldots, x_d\}$ and q'_{d_1} is an intersection point in $L_{d_1} \cap M$. We can then define an evaluation map analogously to before by taking the segment between the outgoing points and reparametrising it by its arc-length to obtain a map

$$\mathrm{ev} : \overline{\mathcal{H}}(q_0, \vec{x}, q_d) \to \Omega(q_0, q_d).$$

Adding all of these breakings to $\mathcal{H}(q_0, \vec{x}, q_d)$, we have:

LEMMA 10.2.8 ([Abo09, Lem. 4.12.]). *The moduli space $\overline{\mathcal{H}}(q_0, \vec{x}, q_d)$ is a compact topological manifold with boundary stratified by smooth manifolds. The codimension 1 strata are the images of the inclusion maps (10.2.10) and (10.2.11).*

In a similar vein to Lemma 10.2.4, we may equip $\overline{\mathcal{H}}(q_0, \vec{x}, q_d)$ with fundamental chains:

LEMMA 10.2.9 ([Abo09, Lem. 4.14.]). *There exists a family of fundamental chains*

$$[\overline{\mathcal{H}}(q_0, \vec{x}, q_d)] \in C_*(\overline{\mathcal{H}}(q_0, \vec{x}, q_d))$$

such that

$$\begin{aligned} \partial[\overline{\mathcal{H}}(q_0, \vec{x}, q_d)] = &\sum_{\substack{0 \leq d_1 \leq d \\ q'_{d_1} \in L_{d_1} \cap M}} (-1)^{\flat} [\overline{\mathcal{H}}(q_0, \vec{x}_1, q'_{d_1})] \times [\overline{\mathcal{H}}(q'_{d_1}, \vec{x}_2, q_d)] \\ &+ \sum_{y \in \mathfrak{X}_H(L_k, L_{k+d_2})} (-1)^{\sharp} [\overline{\mathcal{H}}(q_0, \vec{x}_1, q_d)] \times [\mathcal{R}(y, \vec{x}_2)], \end{aligned} \tag{10.2.12}$$

where

$$\flat = (d_2+1)(|q_0| + |q'_{d_1}| + \sum_{i=1}^{d_1} |x_i|) + d_1 + 1$$
$$\sharp = d_2(|q_0| + \sum_{j=1}^{k+d_2} |x_j|) + d_2(d-k) + k + 1$$

whenever $\overline{\mathcal{R}}(x, \vec{x}_2)$ *is rigid and* x *is the* $(k+1)$*-st element of* $\vec{x}_1$*.*

We can now define the functor:

Definition 10.2.10. *Given the fundamental chains of Lemma 10.2.8, we set*

$$\mathcal{F}^d : \mathrm{CW}_b^*(L_{d-1}, L_d) \otimes \cdots \otimes \mathrm{CW}_b^*(L_1, L_2) \otimes \mathrm{CW}_b^*(L_0, L_1) \to \mathrm{Hom}_*(\mathcal{F}(L_0), \mathcal{F}(L_1))$$
$$[x_d] \otimes \cdots \otimes [x_1] \mapsto \bigoplus_{\substack{q_0 \in L_0 \cap M \\ q_d \in L_d \cap M}} (-1)^{\blacklozenge_d} \mathrm{ev}_*([\overline{\mathcal{H}}(q_0, \vec{x}, q_d)]),$$

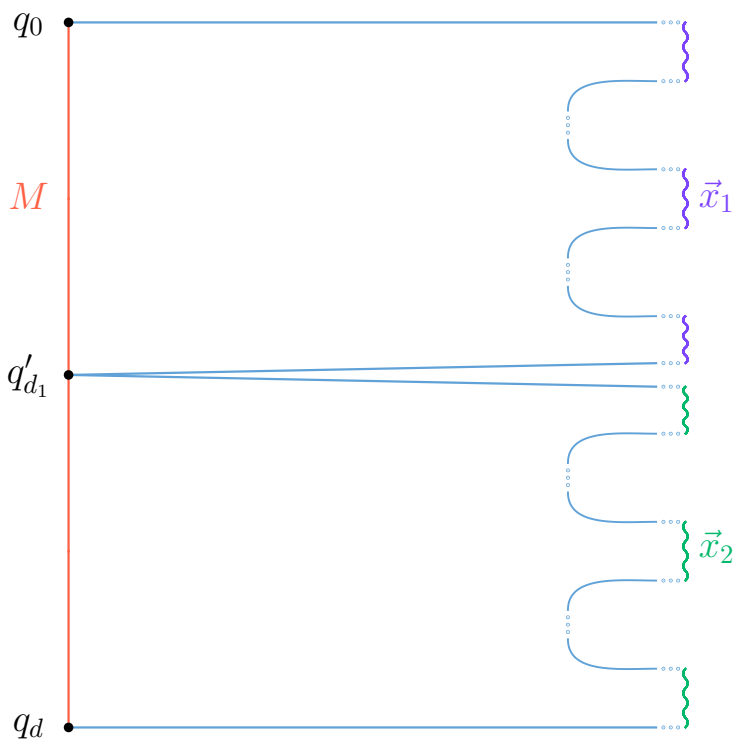
where $\blacklozenge_d := \sum_{k=1}^{d} k|x_k| + (d+1)|q_d| + (|q_0| + d)\dim(\mathcal{H}(q_0, \vec{x}, q_d))$.

Lemma 10.2.11 ([Abo09, Lem. 4.15.]). *The collection of maps* $\mathcal{F}^d$ *satisfy the* A_∞*-equations for functors*

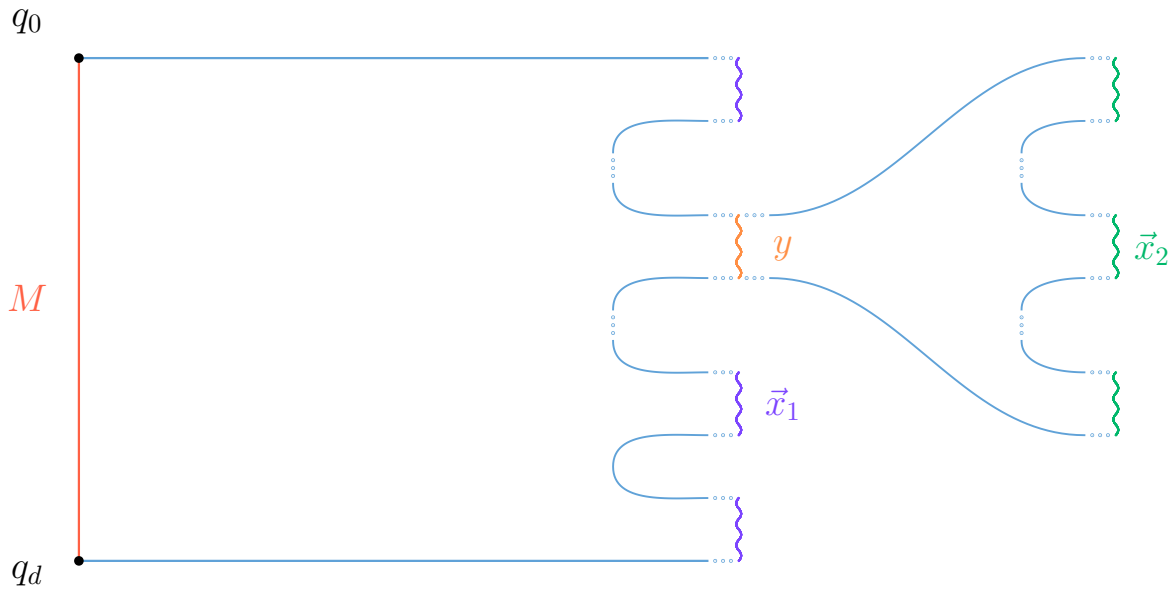
$$\sum_{d_1+d_2=d+1} (-1)^{\blacktriangledown_i} \mathcal{F}^d(\mathrm{id}^{d_1-i} \otimes \mu_{d_2}^F \otimes \mathrm{id}^i) = \mu_1^{\mathrm{Tw}(\mathcal{P})} \mathcal{F}^d + \sum_{d_1+d_2=d} \mu_2^{\mathrm{Tw}(\mathcal{P})}(\mathcal{F}^{d_2}, \mathcal{F}^{d_1}), \tag{10.2.13}$$

where $\blacktriangledown_i$ *was defined in Proposition 9.4.2.*

Proof. The proof will follow similarly to the proof of Lemma 10.2.4 by analysing the boundary components of (10.2.12) one by one. We will skip the discussion of signs. The term $\mu_1^{\mathrm{Tw}(\mathcal{P})} \mathcal{F}^d$ corresponds to the left hand side of Equation (10.2.12) by definition. Next, the term $\sum_{d_1+d_2=d} \mu_2^{\mathrm{Tw}(\mathcal{P})}(\mathcal{F}^{d_2}, \mathcal{F}^{d_1})$ corresponds to the first summation on the RHS of Equation (10.2.12), i.e. products of the type $[\overline{\mathcal{H}}(q_0, \vec{x}_1, q'_{d_1})] \times [\overline{\mathcal{H}}(q'_{d_1}, \vec{x}_2, q_d)]$. Indeed, this component encodes the breaking in between the outgoing points, and thus leads to two instances $\mathcal{F}^{d_1}$ and $\mathcal{F}^{d_2}$ that we need to take the product of. □



The left hand side of Equation (10.2.13) then corresponds to the remaining summation of the RHS of Equation (10.2.12). This type of breaking stems from a breaking occurring away from the outgoing points.



This breaking thus corresponds to compositions of arbitrary higher products in the wrapped Fukaya category with the functors $\mathcal{F}^d$.

11 Return to the Cotangent Bundle

For this final section, let us return to the cotangent bundle. Recall that there exists a decomposition

$$S^*M \times [1, +\infty) \cup D^*M.$$

In [AS12], Abbondandolo and Schwarz proved that there is an isomorphism

$$\Theta : H_*(\Omega(q,q)) \to \mathrm{HW}_b^*(T_q^*M, T_q^*M)$$

using a Morse model on the left hand side.

THEOREM 11.0.1 ([Abo09, Lem. 5.1]). *The linear term*

$$\mathcal{F}^1 : \mathrm{CW}_b^*(T_q^*M, T_q^*M) \to C_*(\Omega(q,q))$$

REMARK 11.0.2. In order to use the results of [AS12] in full generality, one needs to introduce a local system κ_{TM} that depends on whether or not M is orientable, see [Abo09, Rmk. 5.2.].

Since Θ is an isomorphism, it is enough to show that $\mathcal{F}$ is a right-inverse to show Lemma 11.0.1. In order to pass to Morse chains, we follow [AS12] and fix a Morse function $\mathcal{S}$ on

$$\Omega^{1,2}(q,q) := W^{1,2}([0,1],0,1),(M,q,q))$$

the space of maps $[0,1]$ that send both 0 and 1 to q of Sobolev regularity (1,2). Importantly, we assume that $\mathcal{S}$ is the Lagrangian action functional for the Legendre transformation of H. Therefore, the critical points of $\mathcal{S}$ are non-degenerate because of the analogous result for the critical points of $\mathcal{A}$. We thus obtain a Morse-Smale flow on $\Omega^{1,2}(q,q)$ after choosing a sufficiently generic metric. Denote by

$$C_n^{\mathrm{trans}}(\Omega^{1,2}(q,q)) \subset C_n(\Omega^{1,2}(q,q))$$

the subgroup of chains that satisfy:

(H9) the restriction to a boundary stratum of dimension k is smooth near the inverse images of ascending manifolds of codimension k, and intersects the ascending manifolds of codimension larger than k transversally.

We will call such chain satisfying (H9) *weakly transverse*. In particular, this definition implies that a chain of dimension k does not intersect the ascending manifold of critical points of index greater than k. We have the following technical result:

LEMMA 11.0.3 ([Abo09, Lem. 5.3.]). *The inclusion of weakly transverse chains into all chains induces an isomorphism on cohomology. Moreover, for appropriate choices of fundamental chains on the moduli spaces of half-popsicle maps, the map*

$$CF_b^*(T_q^*M; H) \to C_*(\Omega(q,q)) \tag{11.0.1}$$

factors through weakly transverse chains.

The main reason for introducing weakly transverse chains is that the count of negative gradient flow lines starting on a given cell and ending on a critical point of $\mathcal{S}$ defines a chain equivalence

$$C_n^{\text{trans}}(\Omega^{1,2}(q,q)) \to CM_*(\Omega^{1,2}(q,q),\mathcal{S}).$$

If we then compose this map with (11.0.1), we obtain a map

$$CW_b^*(T_q^*M, T_q^*M) \to CM_*(\Omega^{1,2}(q,q),\mathcal{S}),$$

where the right hand side is the Morse complex of the function $\mathcal{S}$. We can now sketch the proof of Lemma 11.0.1.

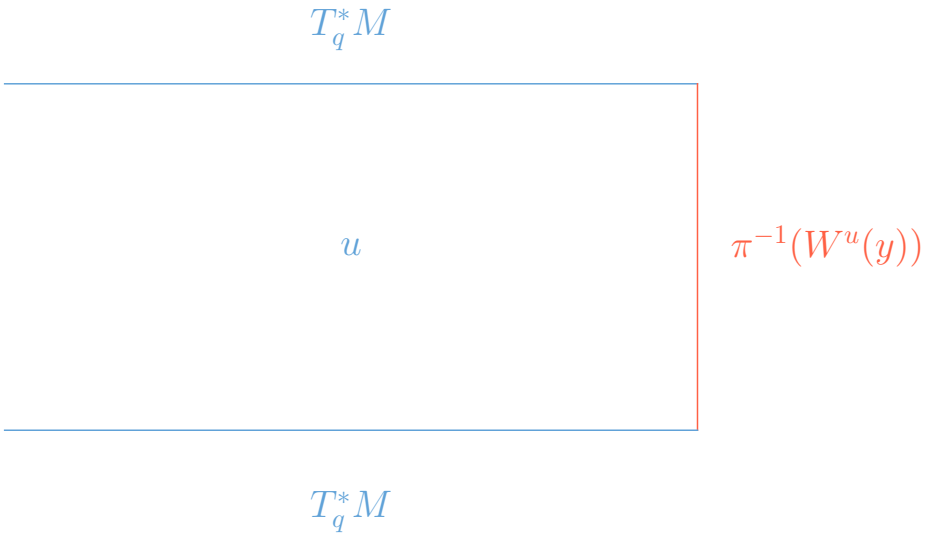
Proof of Thm. 11.0.1. Abbondandolo and Schwarz construct [AS12] in a chain homomorphism

$$\begin{aligned}\Theta : CM_*(\Omega^{1,2}(q,q),\mathcal{S}) &\to CW_b^*(T_q^*M; T_q^*M)\\ y &\mapsto \sum_{|x|=|y|} n^+(y;x)x,\end{aligned}$$

where $n^+(y;x)$ is the count of negative half-strips

$$u : (-\infty, 0) \times [0,1] \to T^*M$$

solving Equation (8.2.3) which converge to x at $-\infty$, and with boundary condition $u|_{(-\infty,0]\times\{0\}} \subset T_q^*M$ and $u|_{(-\infty,0]\times\{1\}} \subset T_q^*M$. In addition these negative half-strips are such that the image of $u(\{0\}\times[0,1])$ under the projection $\pi : T^*M \to M$ is a path on M lying in the descending manifold of y.



We will prove that at the chain level, the composition

$$CM_*(\Omega^{1,2}(q,q),\mathcal{S}) \xrightarrow{\Theta} CW_b^*(T_q^*M, T_q^*M) \xrightarrow{\mathcal{F}^1} CM_*(\Omega^{1,2}(q,q),\mathcal{S})$$

is chain homotopic to the identity. In order to achieve this, we define a parametrised moduli space:

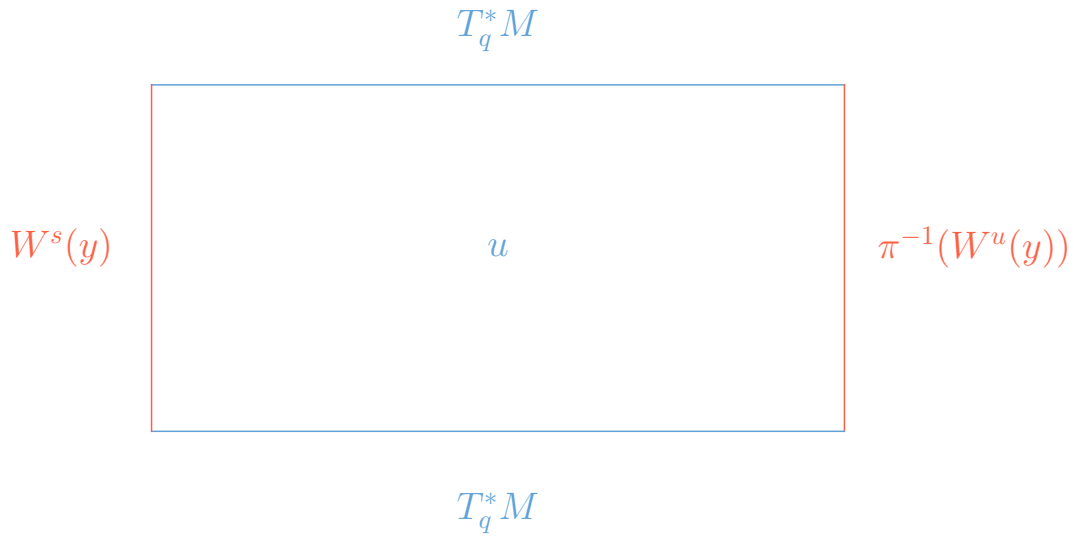
$$\mathcal{C}(y,z;a)$$

consisting of maps

$$v : [0, a] \times [0, 1] \to T^*M$$

that are solutions to Equation (8.2.3), have boundary conditions $u|_{(-\infty,0]\times\{0\}} \subset T^*_qM$ and $u|_{(-\infty,0]\times\{1\}} \subset T^*_qM$, and such that

$$u(\{0\} \times [0, 1]) \subset W^s(z) \text{ and } \pi(u(\{a\} \times [0, 1])) \subset W^u(y).$$



Define

$$\mathcal{C}(y, z) := \bigcup_{a\in[0,\infty)} \mathcal{C}(y, z; a),$$

and let $\overline{\mathcal{C}}(y, z)$ denote its Gromov-compactification, equipped with the Gromov topology. There are now four types of boundary strata. If $a = 0$, then solutions to Equation (8.2.3) must be constant and thus the count of rigid elements of $\mathcal{C}(y, z; 0)$ gives the identity map of $CM_*(\Omega^{1,2}(q, q), \mathcal{S})$. If $a \to \infty$, then a family of solutions $u_a \in \mathcal{C}(y, z; a)$ breaks into two maps u_-, defined on negative semi-strips, and u_+ defined on positive semi-strips. Looking at the definitions, this type of breaking corresponds to $\mathcal{F}^1 \circ \Theta$.

The remaining two breakings occur when energy concentrates at the ends of the strips for finite a. More precisely, one such type of breaking happens if the projection of $u(\{a\} \times [0, 1])$ under π escapes to the ascending manifold of another critical point y'. The other breaking type takes place if the image of $u(\{0\} \times [0, 1])$ converges to the descending manifold of a critical point z'. These boundary strata are in bijective correspondence to

$$\mathcal{T}(y, y') \times \overline{\mathcal{C}}(y'; z) \cup \overline{\mathcal{C}}(y, z') \times \mathcal{T}(z', z),$$

where $\mathcal{T}(y, y')$ is the moduli space of gradient trajectories of $\mathcal{S}$. The count of rigid elements of these moduli spaces defines the differential on $CM_*(\Omega^{1,2}(q, q), \mathcal{S})$. In particular, we have that counting rigid elements of $\overline{\mathcal{C}}(y, z)$ provides a chain homotopy between the identity and $\mathcal{F}^1 \circ \Theta$. □

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